

Summer HSSP 2013 - Relativity

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Lecture 1

Newton, Maxwell, and Great Unifications in Physics

1 Introduction

Welcome to Summer HSSP Relativity class. I am Illan Halpern, a rising senior majoring in Physics and Mathematics. In this class I want to share a little bit on the fascinating topic of Relativity. Let's first take a look at the syllabus to familiarize ourselves with the course structure.

[Logistics Blurb] Feel free at any time to ask me questions!

For many centuries it was believed that space and time were absolute. Space always there, regardless of the observers. And likewise, time always flowing the same way towards the future. Even though this view is seemingly consistent with our everyday experiences, it started to be challenged at the end of the nineteenth century. Our world view was not consistent anymore, and we needed to get rid of some of our old preconceptions, and replace them by some new ideas. And that's when Einstein came in. With special relativity, he was able to reconcile observation and theory. He was able to reconcile Maxwell and Galileo, at the expense of having to modify Newtonian mechanics. Today's lecture will be devoted to understanding the historical context that lead such a huge change in paradigm to be needed. We will review the pre-Newtonian world-view, as well as the revolutions of Newton and Maxwell. But first, what is relativity?

As I mentioned, and will be made more explicit soon, in the beginning of the 20th century, there was a mutual inconsistency if we tried to put Newton, Maxwell and Galileo together. Many people were trying to modify some of these

theories to make the whole consistent, but it was Einstein who succeeded. He preserved Galileo's principle of relativity and Maxwell's electromagnetism unchanged, and modified Newtonian mechanics. In Einsteinian mechanics, known as **Special Relativity**, space and time are no longer absolute, and famous effects such as space-contraction and time dilation arise. The theory of special relativity is what we will spend most of the course talking about.

Soon after he came up with Special Relativity, Einstein realized he had replaced the original problem with a new one. If his theory was correct, then Newtonian gravity could not be. A new theory of gravity was needed, and Einstein was determined to find it. Ten years after publishing Special Relativity, Einstein succeeded in obtaining his theory of gravity, known as General Relativity. General relativity had many experimental successes including correctly explaining the precession of Mercury, as well as correctly predicting the deflection of light. General relativity also had many surprising predictions such as black holes and cosmological evolution. General Relativity will be the topic of the last two lectures.

Einstein's relativity was a great revolution in physics. So were the theories of Maxwell and Newton. What do all of those have in common? They can all be seen as some sort of **unification**. Einstein unified time and space, matter and energy. Maxwell unified the theories of electricity, magnetism and (surprisingly!) optics. Newton unified terrestrial and celestial mechanics. Following the historical order, let's start with Newton.

2 The Newtonian Revolution

Before Newton, the model of world was the one developed in Ancient Times by the likes of Aristotle and Ptolemy. It was the model of celestial spheres. In this model, the first sphere is the sphere of the Earth, where everything is constantly changing, and we terrestrial laws can be applied. The other spheres were the celestial spheres, or the spheres of the skies, where everything is immutable and celestial laws harmony and perfection are applicable. The last of the celestial spheres is the sphere of the fixed stars.

Some years prior to Newton's birth, this view started to be challenged by Galileo, who observed moon craters and moons of Jupiter. However, it was Newton who formulated the first **universal laws**. These include his laws of mechanics as well as his law of gravitation. His law of gravitation is universal because it can be applied to both terrestrial and celestial phenomena. Before we talk about gravity, let's briefly review Newton's laws of motion.

The first law of motion, or the **Law of Inertia** is usually stated as "A body at rest stays at rest, and a body in motion stays in motion, unless acted upon by

an external force. ” Now, what is a force? The second law talks about forces. It relates a the force applied to a body to its acceleration by

$$F = ma = m \frac{d^2x}{dt^2} \quad (1)$$

The third law is the law of action and reaction: it states that action and reaction forces are equal in magnitude and opposite in direction. Now, let’s take a look at what those laws mean. Why do we need the first law? Stated this way it seems that the first law is simply a corollary of the second. Can’t we just plug $F = 0$ in the second law to derive the second? It is certainly true that the second law implies that the acceleration is zero at a body upon which no forces act. However, the first law has an important physical content that is often overlooked.

What was missing in the description of the first law I gave above was the restriction to **inertial frames**. The first law should read “In an *inertial frame*, a body in (uniform) motion stays in (uniform) motion and a body at rest stays at rest.” This law serves as a definition of inertial frames. This is very important since he only claim the validity of his laws in inertial frames. So, by definition, an inertial frame is one in which the first law holds. In such frames, one can apply the second and third laws. If on the other hand, there are no inertial frames, then we can’t apply Newton’s laws of motion. Newton makes a claim about the physical world when he says there is a class of inertial frames. Moreover, he gives an example of an inertial frame: that of the fixed stars.

To apply his laws in an inertial frame, we need to supply a coordinate system. To have an acceleration, we need to have position on time. Once both position and time are well defined, one can measure the acceleration as the second time derivative of position. We can define a coordinate system in Newtonian physics using rulers and a clock. As a body moves, one can look at those measure instruments to determine its position as a function of time, and therefore calculate the acceleration.

Given an inertial frame can we find others? According to Galileo we can. Galileo was the first one to formulate the principle of relativity , according to which the laws of motion are be you at rest or uniformly moving. Imagine you are inside a train cabin with no windows. If there are no windows, then according to Galileo’s principle of relativity, it is impossible to tell if you are at rest or uniform motion. Do Newton’s laws follow this relativity principle? In order for this question to be meaningful, we need to be able to relate the coordinates in both frames: the ground frame and the train frame. Galilean prescription for doing this, known as **Galilean transformation** is

$$\begin{aligned}x' &= x - vt \\t' &= t\end{aligned}\tag{2}$$

The primed coordinates correspond to the train coordinates. v is the velocity of the train as measured from the ground coordinate system. It is assumed that the origins of both systems coincide at time $t = t' = 0$. If that is not the case, then a translation is also needed to related the two coordinate systems. How would the second law read in the train frame?

$$F = m \frac{d^2x'}{dt'^2} = m \frac{d^2(x - vt)}{dt^2} = m \frac{d^2x}{dt^2}\tag{3}$$

So, Newton's second law still holds in the train frame, as long as the force and the mass are the same in both coordinate systems, and the law holds in the ground system. This implies that if the ground system is inertial so is the train system. Using Galileo's Relativity principle we can now start with one inertial system, and get infinitely many: one for every value of v .

Let's look again at the second law. In order for it to be experimentally verifiable a force law is need. Newton provided that by giving his law of gravity. The gravitational force between two bodies of masses m_1 and m_2 , and located a distance r apart from each other, is given by:

$$F = \frac{Gm_1m_2}{r^2}\tag{4}$$

where G is a universal constant known as Newton's constant, or gravitational constant. What was so revolutionary about this law was its universal character. It can both be applied to describe an apple falling from a tree, as well as to describe planetary motion.

3 The Revolution of Maxwell

Let's now skip a couple hundred years, from the 17th to the 19th century. Electric and magnetic phenomena had been observed since ancient times. However, only recently it had started being organized in laws. Maxwell was the one to organize all these laws and give a sense of completeness to the theory. His laws describe electricity and magnetism in terms of fields. His four equations (describing the dynamics of the fields) together with Lorentz force law were supposed to give a complete description of electricity and magnetism. The laws were not created by

Maxwell, though he fixed one of them. He put them together in a unique scheme, and derived the consequences.

Before we can write down Maxwell's equations, we need to know what a field is. A field is a function from space to a target set. A scalar field assigns to each point in space a number. An example of scalar field is temperature: each point in space has a temperature. Vector fields associate to each point in space a vector. Electric and magnetic fields are of this kind. The electric field is denoted by \vec{E} and the magnetic field is denoted by \vec{B} . The over arrow is to remind us that these are vectorial quantities, and therefore possess magnitude and direction.

Maxwell's equations describe how fields evolve in the presence of electric charges and currents. Lorentz force law tells how these fields affect the motion of particles. It is a force law just like Newton's gravity law, and so the Lorentz force can be plugged into Newton's second law to give an acceleration. Lorentz force law is:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}, \quad (5)$$

where q is the electric charge, and v is the velocity of the particle. The $\vec{v} \times \vec{B}$ term is the cross product between the vectors \vec{v} and \vec{B} . The cross product of vectors \vec{X} and \vec{Y} is the vector having magnitude $|\vec{X}||\vec{Y}|\sin\theta$, where θ is the angle between \vec{X} and \vec{Y} , and direction given by the right-hand rule.

The four Maxwell's laws are: Gauss law, no-magnetic monopoles law, Ampere's law, Faraday's laws. Gauss law describes how charges produce an electric field. The analog of Gauss law for magnetism states that there are no "magnetic charges," or monopoles. So where then does \vec{B} come from? Ampere's law provides the answer: the magnetic field is produced by currents, or moving electric charges. Faraday law started unifying electricity and magnetism by showing how magnetic field can induce an electric field. Maxwell completed this unification by fixing Ampere's law to show that an electric field can also induce a magnetic field.

3.1 Gauss Law and No Magnetic Monopoles

For a static charge distribution Gauss law is equivalent to Coulomb's law. Coulomb's law is:

$$\vec{E} = \frac{kQ}{r^2}\hat{r}, \quad (6)$$

where \vec{r} is the relative position vector, $r = |\vec{r}|$ and $\hat{r} = \frac{\vec{r}}{r}$.

Gauss Law states the same thing in a different way. It relates the electric flux on a surface to the charge enclosed. We can represent Coulomb's law graphically using field lines. In this schematic representation, the direction of the lines represent the direction of the electric field and their density represents the magnitude of the

electric field. The electric flux is a measure of the number of field lines crossing a surface. If we imagine an infinitesimal surface with normal area vector $d\vec{a}$, then the electric flux on the surface is given by $\Phi_E = \vec{E} \cdot d\vec{a}$. For an extended surface, the total flux is obtained by integrating the flux at each infinitesimal area element.

$$\Phi_E = \int_S \vec{E} \cdot d\vec{a} \quad (7)$$

Gauss law says that the electric flux on a closed surface is proportional to the charge enclosed by it.

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{a} = \frac{Q}{\epsilon_0} \quad (8)$$

The constant ϵ_0 is called the vacuum permittivity. The circle on the integral in to remind us that in order for Gauss Law to be applicable, the integral must be done over a closed surface.

The magnetic flux Φ_B is defined analogously by simply replacing \vec{E} with \vec{B} in equation (7). The analogous of Gauss Law for magnetism is

$$\Phi_B = \oint_S \vec{B} \cdot d\vec{a} = 0 \quad (9)$$

This is the statement that there are no magnetic monopoles, or magnetic charges.

3.2 Ampere's Law and Faraday's Law

The source of magnetic field are electric currents. The current in a wire is defined as $I = \frac{dQ}{dt}$, the rate at which charge passes through a normal cross-sectional area of the wire. For a static charge distribution, we have seen that Gauss Law is equivalent to Coulomb's law. In magnetism, we have the same thing: for a stationary current, Ampere's law is equivalent to the Biot-Savart Law. Ampere's law is:

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I \quad (10)$$

This law state that the integral of the magnetic field along a closed curve (a loop) is proportional to the current that crosses the area enclosed by the loop. The constant of proportionality μ_0 is referred as the vacuum permeability. This is in a way analogous to the Gauss Law. Gauss Law can be used to calculated the electric field, while Ampere's Law can be used to calculate the magnetic field. The analogous of Coulomb's law would be Biot-Savart Law:

$$\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2} \quad (11)$$

Faraday's law describes how a time-varying magnetic field induces an electric field. The effect of this law can be seen when one moves a wire next to a magnet, a magnet next to a wire, or changes the intensity of the magnet keeping both fixed. In all three cases, there will appear a current flowing through the wire. The current is related to the magnetic flux across the loop of wire by

$$I = \frac{1}{R} \frac{d\Phi_B}{dt}, \quad (12)$$

where R is called the resistance of the loop. What causes the electron in the current to move? It is an electric field. So, any other way of understanding magnetic induction is through Faraday's law:

$$\oint \vec{E} d\vec{l} = - \frac{d\Phi_B}{dt} \quad (13)$$

This law started to unify electricity and magnetism since it shows how the magnetic field can induce an electric field.

3.3 Maxwell's Equations and Electromagnetic Waves

Maxwell noticed that Ampere's law was not well-defined. Given a loop, there is an ambiguity in the choice of enclosed surface. The choice of different surfaces, can lead to different fluxes, giving different predictions. Maxwell noted that if another term was added to the equation, then the sum would be independent of the surface chosen. With this fix, Ampere's law becomes:

$$\oint \vec{B} d\vec{l} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \quad (14)$$

The set of equations (8), (9), (13) and (14) are called Maxwell's equations. The new term added to equation (10) in equation (14) is very important conceptually. It shows that not only the magnetic field can induce an electric field, as in Faraday's law, but also an electric field can induce a magnetic field. This suggests that electromagnetic fields can self propagate. And indeed, Maxwell found a vacuum solution of his equation that takes the form of a propagating wave. From his equation he was also able to derive that the speed of such an electromagnetic wave would be $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$. The values of the constants μ_0 and ϵ_0 had already been measured experimentally, so he just plugged the numbers to get the speed of an electromagnetic wave. To his awe, the calculated value was $3 \times 10^8 \text{ m/s}$, which

happened to be the measured speed of light! He was forced to conclude that light is but a disturbance of the electromagnetic medium. Light is an electromagnetic wave! In attempting to unify electricity and magnetism, Maxwell had also unified them to optics!

So, this was the world that Einstein saw himself in. As we will see next time, it is a logical impossibility for all of Galileo's principle of relativity, Newtonian Mechanics, and Maxwellian Electromagnetism to be right. To solve this problem, Einstein created his theory of Special Relativity.