11483: Introduction to Modern Physics
Lecture-3

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1 Special Relativity : Dynamics

1.1 The Problem with Newtonian Mechanics

Consider an object of mass m_0 initially at rest. We apply a constant force $F\hat{x}$ on the object starting at $t = 0$. From Newton's second law, the acceleration of the particle is $F/m\hat{x}$ so its speed at any time t is Ft/m.

Clearly, there is no upper bound on this speed. If we apply the force for long enough, we can achieve an arbitrarily large speed. This would contradict the fact that c is the maximal allowed speed.

1.2 Relativistic Mass

In relativity, the mass and energy are related via the famous equation

$$E = mc^2 \tag{1}$$

Here, m is the relativistic mass of the object. The relativistic mass is a redundant quantity because it is simply E/c^2.

Since energy and correspondingly relativistic mass depend on the object's speed, Newton's second law in relativity is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{2}$$

where $\mathbf{p}$ is the object's momentum.

2 4-Vectors

Since the Lorentz transformations mix up space and time we can combine them into a single four component object called a 4-vector. Thus, we have the space-time 4-vector $\tilde{s} = (ct, x, y, z) = (ct, \mathbf{x})$ where $\mathbf{x}$ is the distance vector.

Given an inertial reference frame F and another inertial reference frame F' moving at velocity $u\hat{x}$ with respect to F the spacetime 4-vectors in these two frames are related via the Lorentz transformation,

$$
\begin{aligned}
ct' &= \frac{ct - u/c\,x}{\sqrt{1-u^2/c^2}} \\
x' &= \frac{x - u/c\,ct}{\sqrt{1-u^2/c^2}} \\
y' &= y \\
z' &= z
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
ct &= \frac{ct' + u/c\,x'}{\sqrt{1-u^2/c^2}} \\
x &= \frac{x' + u/c\,ct'}{\sqrt{1-u^2/c^2}} \\
y &= y' \\
z &= z'
\end{aligned}
\tag{4}
$$

2.1 Proper Time and 4-Velocity

We define the proper time as

$$
\Delta\tau^2 = -\frac{\Delta s^2}{c^2} = \Delta t^2 - \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{c^2} \tag{5}
$$

Note that this is consistent with the fact that $\Delta\tau = \Delta t$ for collocated events. Also, proper time is clearly a frame invariant.

Now we define the 4-velocity as rate of change of spacetime 4-vector with respect to proper time.

$$
\begin{aligned}
\tilde{v} &= \frac{\Delta\tilde{s}}{\Delta\tau} \\
&= \left(c\frac{\Delta t}{\Delta\tau}, \frac{\Delta x}{\Delta\tau}, \frac{\Delta y}{\Delta\tau}, \frac{\Delta z}{\Delta\tau}\right)
\end{aligned}
\tag{6}
$$

For an object moving with speed v,

$$
\frac{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}}{\Delta t} = v \tag{7}
$$

$$
\Delta\tau^2 = (1 - v^2/c^2)\Delta t^2 = \frac{\Delta t^2}{\gamma(v)^2} \tag{8}
$$

$$\begin{aligned}\tilde{v} &= (\gamma(v)c, \gamma(v)v_x, \gamma(v)v_y, \gamma(v)v_z) \\ &= (\gamma(v)c, \gamma(v)\mathbf{v})\end{aligned} \tag{9}$$

where $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity of the object.

2.2 Relativistic Energy and Momentum

Similar to the spacetime 4-vector, we have the momentum 4-vector $\tilde{p} = (E/c, p_x, p_y, p_z) = (E/c, \mathbf{p})$ where E is the energy of the object and $\mathbf{p} = (p_x, p_y, p_z)$ is the momentum.

Now consider an object of mass m_0 at rest in F'. The energy of this object is $E' = m_0c^2$ coming from its rest mass and the momentum is $\mathbf{p}' = (0, 0, 0)$ as the object is at rest. Thus, using the Lorentz transformations

$$\begin{aligned}E/c &= \frac{E'/c + u/c\, p'_x}{\sqrt{1-u^2/c^2}} = \frac{m_0c}{\sqrt{1-u^2/c^2}} \\ p_x &= \frac{p'_x + u/c\, E'/c}{\sqrt{1-u^2/c^2}} = \frac{m_0u}{\sqrt{1-u^2/c^2}} \\ p_y &= p'_y = 0 \\ p_z &= p'_z = 0\end{aligned} \tag{10}$$

These are the energy and momentum of an object with velocity $u\hat{x}$.

Thus, for an object with rest mass m_0 and velocity $\mathbf{v}$,

$$E = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} = \gamma(v)m_0c^2 \tag{11}$$

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}} = \gamma(v)m_0 \tag{12}$$

$$\mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1-v^2/c^2}} = \gamma(v)m_0\mathbf{v} \tag{13}$$

Clearly, 4-momentum and 4-velocity are related via the rest mass as

$$\tilde{p} = m_0\tilde{v} \tag{14}$$

similar to its classical analogue.

The following plots show the energy and momentum as a function of speed. The non-relativistic analogues are also shown for comparison

2.3 Relativistic Dispersion Relation

A dispersion relation is a relationship between the energy and momentum of an object. Recall that the photon dispersion relation is

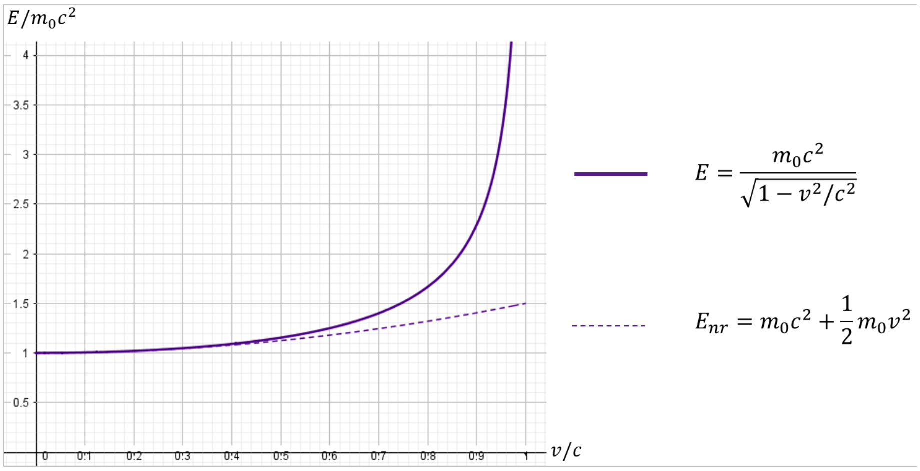

Figure 1: Plot of E/m_0c^2 vs. v/c.

$$E = pc \tag{15}$$

Similarly we can derive the relativistic dispersion relation,

$$\begin{aligned}
E^2 &= \frac{m_0^2c^4}{1-u^2/c^2} && (16)\\
&= \frac{m_0^2c^4[(1-u^2/c^2)+u^2/c^2]}{1-u^2/c^2} && (17)\\
&= m_0^2c^4 + \frac{m_0^2u^2c^2}{1-u^2/c^2} && (18)\\
E^2 &= m_0^2c^4 + p^2c^2 && (19)
\end{aligned}$$

In the low momentum limit i.e. $p \ll m_0c$,

$$\begin{aligned}
E &= \sqrt{m_0^2c^4 + p^2c^2} && (20)\\
&= m_0c^2\sqrt{1+\frac{p^2}{m_0^2c^2}} && (21)\\
&\approx m_0c^2\left(1+\frac{1}{2}\frac{p^2}{m_0^2c^2}\right) && (22)\\
E &\approx m_0c^2 + \frac{p^2}{2m_0} && (23)
\end{aligned}$$

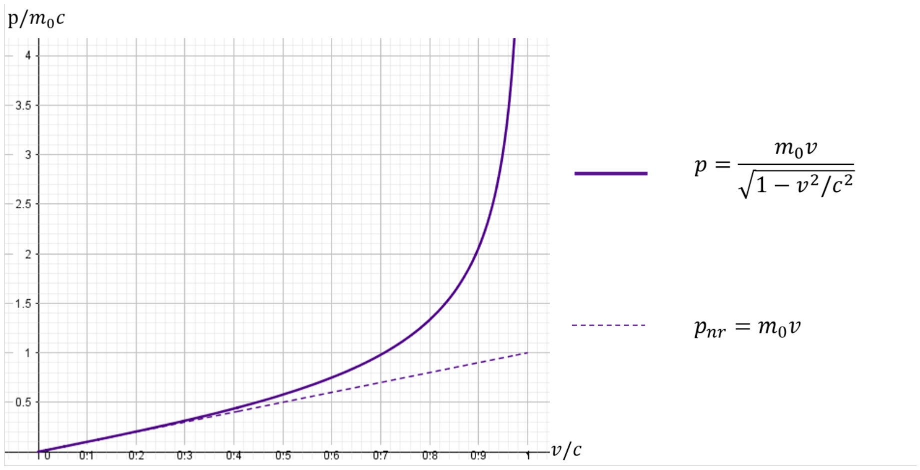

Figure 2: Plot of p/m_0c vs. v/c.

which is the classical expression for energy.

In the high momentum limit i.e. $p \gg m_0 c$,

$$E \approx pc \tag{24}$$

which is the photon dispersion relation.

3 General Relativity : Introduction

3.1 Equivalence Principle

The equivalence principle states that it is impossible to distinguish between

(i) an accelerated reference frame, with uniform acceleration $\mathbf{a}$, and

(ii) a stationary reference frame in a uniform gravitational field, $\mathbf{g} = -\mathbf{a}$. using any physical experiment.

Alternatively, inertial mass equals gravitational mass. Inertial mass is the mass that shows up in Newtonian mechanics

$$\mathbf{F} = m_i \mathbf{a} \tag{25}$$

Gravitational mass is the mass in Newton's law of gravitation

$$\mathbf{F} = \frac{GMm_g}{r^2}\hat{r} \tag{26}$$

Thus, equivalence principle is

$$m_i = m_g \tag{27}$$

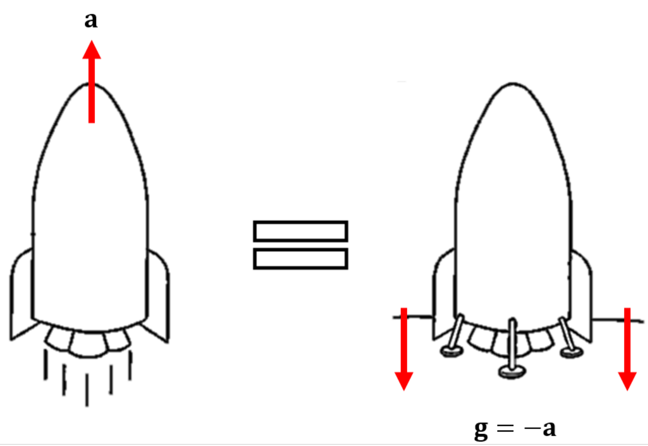

Figure 3: Equivalence principle states that the two scenarios are equivalent.

This allows us to deal with gravitational systems using a locally comoving reference frame i.e. an inertial reference frame in which the object of interest is instantaneously at rest and treating the gravitational field as an acceleration.

Furthermore, any free-falling reference frame is equivalent to an inertial reference frame because the effects of gravitation are cancelled out by the effects of acceleration as the reference frame is in free-fall.

3.2 Gravitational Redshift

The energy of a photon is

$$E = hf = hc/\lambda \tag{28}$$

Now consider a photon travelling upwards in a gravitational field. Energy conservation requires that the energy of the photon decreases as it goes up. Thus, the photon has a lower frequency or higher wavelength at higher gravitational potential. This phenomenon is called gravitational redshift. Needless to say, a photons travelling downwards in a gravitational field is blueshifted.

In case of the field of a point mass M, if the photon has frequency f_0 at a distance r_0 from the mass then its frequency at distance r is

$$f = f_0 \sqrt{\frac{1 - 2GM/r_0c^2}{1 - 2GM/rc^2}} \tag{29}$$

In the weak gravity limit,

$$f \approx f_0 \left(1 - \frac{GM}{r_0 c^2} + \frac{GM}{rc^2}\right) \tag{30}$$

$$hf_0 - hf \approx GM\frac{hf_0}{c^2}\left(\frac{1}{r_0} - \frac{1}{r}\right) \tag{31}$$

$$E_0 - E \approx GM\frac{E_0}{c^2}\left(\frac{1}{r_0} - \frac{1}{r}\right) \tag{32}$$

This is consistent with conservation of energy.

3.3 Gravitational Time Dilation

A decrease in frequency implies an increase in time period as $T = 1/f$, so

$$t = t_0\sqrt{\frac{1 - 2GM/rc^2}{1 - 2GM/r_0c^2}} \tag{33}$$

Set $r_0 = \infty$ so that t_0 is the time for an observer in an inertial reference frame because there is no gravitational force on this observer. Thus, t_0 is called the coordinate time.

$$t = t_0\sqrt{1 - \frac{2GM}{rc^2}} \tag{34}$$

Clearly, $t < t_0$ so the time between two events measured by a clock in a gravitational field is less than the coordinate time. Thus, clocks located at lower gravitational potential run slower. This phenomenon is called gravitational time dilation.

3.4 Gravitation and Spacetime

In classical mechanics, gravity is treated as a force that acts on objects present in space. However in general relativity, gravity influences the very nature of spacetime. This means that the spacetime interval between events can be changed by a gravitational field. Thus, distances in general relativity becomes a gravity-dependent notion.

The presence on any massive object wraps spacetime around itself. The more massive the object, the more stronger is this wrapping. In a curved spacetime, the trajectory of a free object is non-linear.

These curved spacetimes are characterized by the curvature , Ω. Curvature can be seen as the extent to which a spacetime is curved with curvature of a plane being zero. Spherical surfaces have a positive curvature with smaller spheres having larger curvature.

The curvature of a spacetime can be characterized by the sum of interior angles of a triangle drawn on that spacetime.

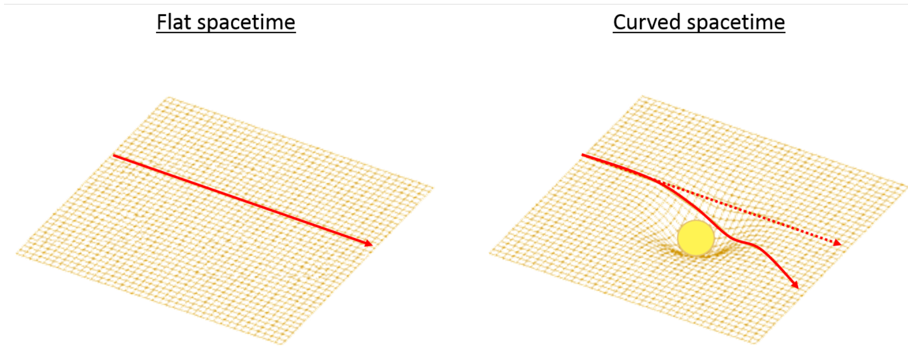

Figure 4: Spacetime curves due to massive objects.

- Sum of interior angles $= 180^\circ \implies \Omega = 0$ i.e. zero curvature or flat spacetime.
- Sum of interior angles $> 180^\circ \implies \Omega > 0$ i.e. positive curvature.
- Sum of interior angles $< 180^\circ \implies \Omega < 0$ i.e. negative curvature.

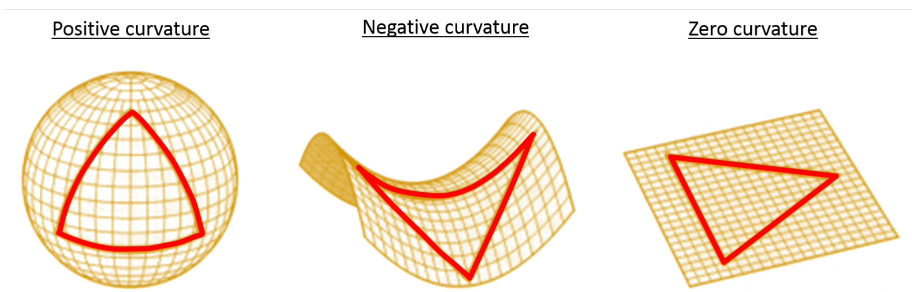

Figure 5: Types of curvature.

To summarize: spacetime tells matter how to move; matter tells spacetime how to curve.