Solve Wave Equation from Scratch [2013 HSSP] [1]

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I. COURSE INFO

Topics

Date	
07/07	Complex number, Cauchy-Riemann Equation, Cauchy Theorem
14/07	Cauchy Theorem, Cauchy Integral Formula, ODE
21/07	Cauchy integral Theorem, Fourier Transform
28/07	Fourier Series, Fourier Transform, 1st order PDE
04/08	1st order PDE, Laplace Transform
11/08	Operator, ODE, Green's Function
18/08	Green's function of Wave Equation, Solution to Wave Equation

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II. Q & A

07/07
1. Do not know much about multivariable calculus
-This course only involves the very basics of multivariable calculus, such as partial derivative; there won't be Divergence theorem, Green's theorem etc. There are lots of materials accessible on the Internet for multivariable calculus, such as MIT's OCW and Youtube videos. Besides, ask questions during lectures whenever it's not clear to you.

14/07
2. Don't understand Double pole
-See the first two sections (A and B) of notes.

21/07
3. Why do some integrals go to zero?
There are two ways we can conclude an integral is equal to zero: using Cauchy theorem and not using Cauchy theorem. For the second case, there are two possibilities: 1. Very large contour - change to polar coordinates and take R to infinity; if the power of R is larger in the denominator the integral is zero; 2. Very small contour - change to polar coordinates and take R to 0; if the power of R is larger in the numerator the integral is zero
I presume you are complaining about those ones in the example of branch cuts. In that case you have a very small contour. The power of R is 1 in the numerator, the power of R is 1/2 in the denominator because 1+x and 1-x are

[1] The materials of this document will be cumulative because I don't want to create separate Latex files and you shouldn't complain.

both approximated to 1. If it is x instead of 1+x the integral will blow up.

4. Branch cuts and branch points

Branch points are points that don't have unique values due to rotations. The purpose of branch cuts is to prevent ambiguities; branch cuts are in walls that contours cannot cross. If the rotation is never completed or the contour is never closed, there won't be any ambiguity. With the purpose of branch cuts in mind, it might be easier to think of how to make them. Branch cut can be finite or infinite as long as it serves its purpose. If there's only one branch point, it is always infinity; if there are two or three branch points the branch cut might be finite. However, we can still close the contour if there is an infinite branch point. Regrading Cauchy integrals, the integrals that enclose branch points are not well defined, so it cannot be evaluated; we must find a closed contour that doesn't have any branch point.

5. Fourier Transform

We will talk more about it next time. See below (D) for physics-related topics.

28/07

6. Proof of Inversion theorem.

We assume that inversion theorem of Fourier transform holds if we can get the same function back using inversion. It is fair as a mathematical assumption (maybe not as a philosophical assumption). There are a few steps in the proof of inversion theorem: change the limits of integral from infinities to finite end points, apply the inversion theorem to $\tilde{f}(k)$, apply the Fourier transform to $f(x)$ and take the finite end points to infinities. Following this scheme, you should be able to prove the inverse transform.

7. Regarding Cauchy integral and Fourier transform:

1. Integrating in complex plane enables you to evaluate integrals (by hand) which you cannot do or have a very hard time doing using calculus. 2. When doing Fourier transform whose nature is an integral, you might encounter integrals that is easy to evaluate in complex plane and difficult in calculus. 3. With the help of Mathematica or other softwares, you can evaluate almost all integrals without knowing Cauchy theorem, but almost all is not all, I've done, in my problem sets for this class, many problems (set up by a skilful instructor) that are not solved by Mathematica. So I think it is still good to know; it's a nice exercise for brain as well.

8. Fourier Series [I am not sure if I understand your questions]

The idea of Fourier series is to write a function in terms of sines and cosines or equivalently complex exponentials. Only differentiable functions with finite end points can be written in this form; a counterexample is e^x. Fortunately, most functions that are encountered in science and engineering have such properties. All the definitions are quite straightforward. I guess the confusing part is the differential operator story; we might as well discuss it in class if time permits.

9. Partial Differential Equation

We will continue next time.

10. Convolution

Convolution is an operation for two or more functions. We will talk more about it when it comes to Laplace transform. It is also very common in probability theory, where you might want to convolve two probability distributions to obtain the total probability distribution.

04/08

11. Partial differential equation's Characteristic curves

Along the characteristic curves, the function either is constant for homogeneous PDE, or has a constant rate of change for inhomogeneous rate of change.

12. What is inversion theorem of Laplace transform?

The function f(x), for example, is the description of an object in x space; the Laplace transform of it is the description of the SAME object in a different language. The reason to perform Laplace, which is the also the reason to apply any transform, is that some calculation is simpler in one space than the other. But after you obtain an expression in terms of s, you might want to transform it back, into a space where you can interoperate more easily, or so to speak, into a language which you are more familiar with. Then you need to use the inversion theorem.

13. Connection ODE and Laplace Transform

Laplace transform is generic method of solving Ordinary Differential Equations. The reason to perform Laplace, which is the also the reason to apply any transform, is to simplify the calculation. The Laplace transform of derivatives can be related to the Laplace transform of the function itself (through integration by parting); this allows you to calculate the explicit express of the Laplace transform of the function, from which you can obtain the solution by applying inversion theorem.

11/08
14. Definition of Green's function
We haven't got to the point of defining Green's function (but it is always a good idea to know the definition), so I will say more on it next time. It is the function that yields a Dirac delta function when the PDE operator acts on in. For Wave Equation, for example, the operator is $\partial_t - \nabla^2$, Green's function is the function such that,

$$(\partial_t - \nabla^2)G(x - x') = \delta^{(4)}(x - x')]$$

15. Delta function
I always refer to Dirac Delta when I say Delta function. It has infinitesimal width and infinite height, but its integral is one.

16. ∇^2, Laplace or Laplacian
I found out that Wikipedia calls it Laplace operator or Laplacian and Wolfram calls it Laplacian.

18/08
17. What is G?
The Green's Function

18.$U(\vec{k}, t)$ into formula?
Plug $U(\vec{k}, t)$ into the Wave Equation and you should obtain: $(-k^2 + \partial_t^2)U(\vec{k}, t) = 0$. We did this step last week; refer to your note for detail.

19. G_1, G_2 substitution
In order to obtain the solution of wave equation, we substitute the Green's functions into:

$$u(\vec{x}, t) = \int d^n\vec{x'}[G_1(\vec{x} - \vec{x'})a(\vec{x'}) + G_2(\vec{x} - \vec{x'})b(\vec{x'})]$$

Before substituting, make sure you have the right Green's function.

20. Motion in String Theory
A string is still a string in String Theory, so it moves like a string. However, in order to have a relativistic string, we need to incorporate time coordinate.

III. SUPPLEMENTARY NOTES

Chances are there will be typos and mistakes since the notes are created by me. Please let me know if there's anything wrong or confusing.

A. Taylor Series and Laurent Series

The idea of Taylor series is that all analytic functions (i.e. functions whose nth derivative exists) can be expressed as the sum of polynomials. The coefficients are determined by the derivative at a particular point. The function is written as,

$$f(z) = f(z_0) + \frac{f^{'}(z_0)}{1!}(z - z_0) + \frac{f^{''}(z_0)}{2!}(z - z_0)^2 + ... + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + ... \qquad (1)$$

To save some ink, we rename the coefficients,

$$f(z) = a_0 + a_1(z - z_0) + a_2 z - z_0)^2 + ... + a_n(z - z_0)^n + ... \qquad (2)$$

When the function is no longer analytic, it cannot be expressed as the sum of nice polynomials because it has singularities. So we think of extending Taylor series into a more generic power series, namely, Laurent series. If f(z) has a singularity at z_0 and it is a simple pole, then f(z) can be expressed as,

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2 z - z_0)^2 + ... + a_n(z - z_0)^n + ... \tag{3}$$

If f(z) has a double pole at z_0,

$$f(z) = \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ... + a_n(z - z_0)^n + ... \tag{4}$$

B. Residual and Double Pole

Residual is simple a_{-1} of a Laurent series. From this definition, we know that it depends on the function, the contour and the singularities enclosed.
In order to find a_{-1}, we multiply both sides by $(z - z_0)$ and take the limit of z goes to z_0,

$$\lim_{z \to z_0} f(z)(z - z_0) = \lim_{z \to z_0} (a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + a_2(z - z_0)^3 + ... + a_n(z - z_0)^{n+1} + ...) \tag{5}$$

This is what happens when we evaluate the integral.
When z_0 is a double pole, the same trick doesn't work because of the a_{-2} term,

$$\lim_{z \to z_0} f(z)(z - z_0) = \lim_{z \to z_0} (fraca_{-2}z - z_0 + a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + a_2(z - z_0)^3 + ... + a_n(z - z_0)^{n+1} + ...) \tag{6}$$

In order to obtain a_{-1}, we do instead,

$$\lim_{z \to z_0} \frac{d}{dz}[f(z)(z - z_0)^2] = \lim_{z \to z_0} \frac{d}{dz}(a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + a_1(z - z_0)^3 + ... + a_n(z - z_0)^{n+2} + ...) \tag{7}$$

Clearly, a_{-1} is the only thing left after this.
If a function contains $\frac{1}{(z - z_0)^3}$ or even worse, we multiply whatever that sends the denominator to 1, and take derivatives. Always keep in mind what you want to obtain before manipulating the function.

C. Principle Value of an Integral

Principle value method is a way to evaluate integral. It is neither smatter nor simpler than Cauchy integral formula in my opinion. I was taught this because I "might need to communicate with the rest of world" . The difference before principle value and Cauchy method is that there is an extra step in principle value before applying Cauchy integral formula.
The idea of principle value is that we jump over the singularities when evaluating an integral. This extra step which presumably makes it more straightforward. Principle value of an integral is the same as the integral's value when the integral converges, yet it exists even if the integral diverges (i.e. goes to infinity).

$$P = \lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} f(z)dz + \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} f(z)dz \tag{8}$$

The integral from $-\epsilon$ to ϵ goes to zero as ϵ goes to zero.

Example:

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$$

In this example I will only demonstrate the extra step of Principle value method to show how principle value method is connected to Cauchy theorem. The rest of the example which is the Cauchy method will be completed in class.
Solution:

First note, the principle value of this integral is the true value of the integral because the function is convergent. There is a singularity at 0, but the function doesn't go to infinity there. So, $I = P\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$. $\frac{\sin x}{x}$ is the imaginary part of $\frac{e^{ix}}{x}$ by Euler's equation. Now define $J = P\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$ and I = Im(J).

$$J = \lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} \frac{e^{ix}}{x} dx + \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{e^{ix}}{x} dx$$

This brings us to the starting point of Cauchy theorem.

D. Fourier Transform

Heisenberg uncertainty principle (position and momentum): $\Delta x \Delta p > \hbar/2$ (there is also an energy and time uncertainty principle). It says momentum and position of an object cannot be determined with 100% certainty simultaneously. Mathematically, the Fourier Transform of Dirac delta is a sinusoidal curve. Using physical intuition, we know this makes sense because when position is very well defined (Dirac delta), its momentum spreads out. This is one example showing math and physics agree with each other. $f(x)$ and $\tilde{f}(k)$ describe the same object but in two different languages.
An interesting subtlety concerning Fourier transform is that nothing is infinity is physical world, so $f(x)$ and $\tilde{f}(k)$ are not equivalent unless they are infinite. This somehow seconds that transform is not the same as a change of variable. However, it is hard to think at infinity; this is easier to see if you think of x and p as matrices. In non-commutative algebra (whatever that means), Heisenberg principle is $[x, p] = i\hbar I$. The dimensions of trace are not equal on the two sides unless x and p have infinite dimensions.

tl; dr (totally your loss if you didn't.) Math goes hand in hand with physics.

E. Fourier series

Applet: http://mathlets.org/mathlets/ [Courtesy of MIT 18.03 instructors]

F. Useful Tools

Latex - Allows you to type everything and anything; Easy to format; used in (almost) all science and math journals; Free

MATLAB - Good for matrix and differential equations (yet not the best and sometimes crashes); C++ syntax; Popular for data analysis (popular is not necessarily the best); not free

Mathematica - Good for symbolic manipulations; Creates beautiful graphs; has its own syntax, but it's super straightforward; Help Centre actually helps; not free

MATLAB and Mathematica are not free, actually quite pricey, but chances are your college will pay for you. They doesn't help in either high school math or Olympiad. I didn't use any of these in high school, but they are extremely helpful to me in college.