Radiation, Antennas, and Einstein Relativity: What They Won't Tell You in AP Physics*

Christian Ferko

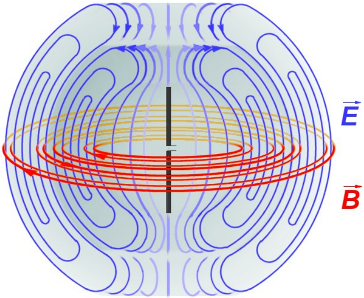
November 15, 2020

*Warning: parts of this talk will move fast and you may not understand all of the details. That's okay! My goal is to convey a *flavor* of the subject and a few of the main ideas. Don't worry if some things are confusing.

Introduction: antennas and nudging charges.

Transmitting and receiving.

We begin with a very down-to-earth question: how does a radio station transmit music to you in the car?



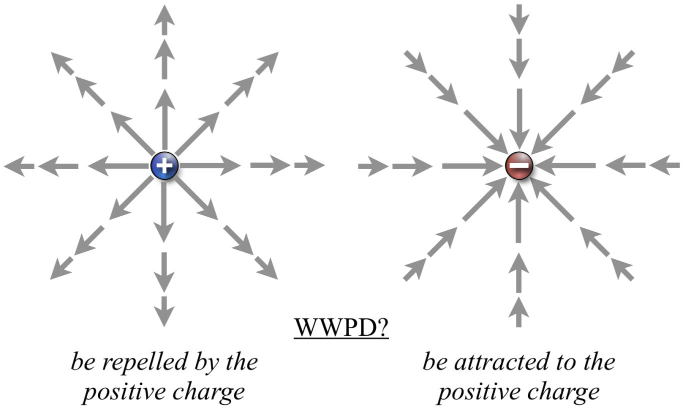
$\longrightarrow$

Short answer: the transmitter shakes some electrons, which produces radiation. In this talk we'll try to understand those words.

What would a positive charge do?

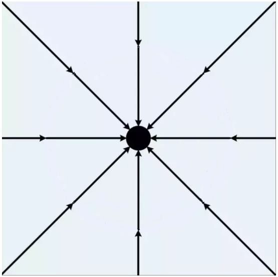
Before diving into details, let's look at some pictures for intuition.

Positive charges move away from other positive charges, and toward negative charges. Visualize as **electric field**: a bunch of arrows that point in the direction a positive charge placed at that location would move.

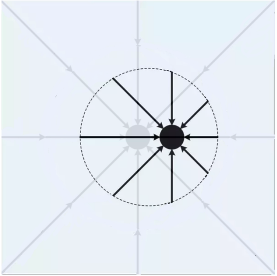


Nudging a charge.

Suppose I grab an electron, nudge it to the right, and let go.



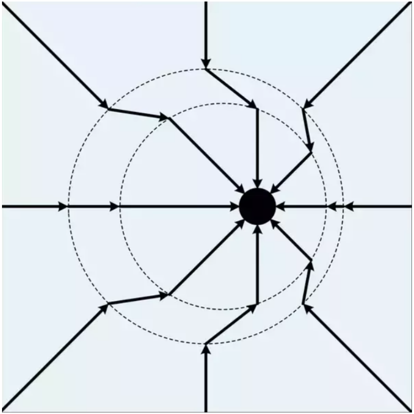
$\longrightarrow$



An important principle of physics is **locality**: an object can only influence nearby objects. If we move the charge, another charge far away (on the moon) cannot *instantaneously* know that the field has changed.

Radiation stitches old to new.

Since the electric field cannot change immediately, far away from the electron, the electric field must point to the *old location of the charge*.



If we assume the electric field is smooth, there must be an in-between region which “stiches” between the old and new fields. This is **radiation**.

Roadmap.

Punchline: Locality means that nudging a charge creates a shell moving outward which updates the electric field to the charge's new location.

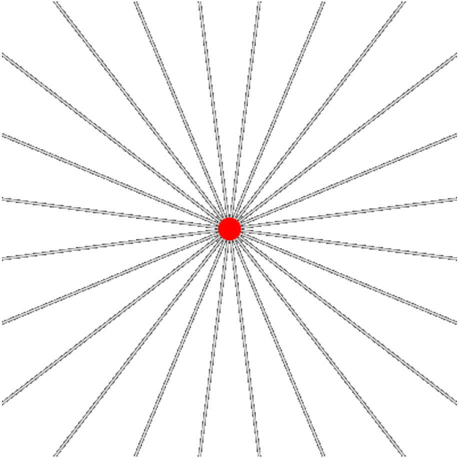
The rest of the talk will explore the details and consequences of this statement. Here is the plan:

- ☑ Introduction: antennas and nudging charges.
- ☐ Act 1: putting a speed on locality.
- ☐ Act 2: a tour of Maxwell's equations.
- ☐ Interlude: dipole antennas.
- ☐ Act 3: Einstein and the weird constancy of c.
- ☐ Epilogue: magnetism and principles of theoretical physics.

Act 1: putting a speed on locality.

A propagation speed.

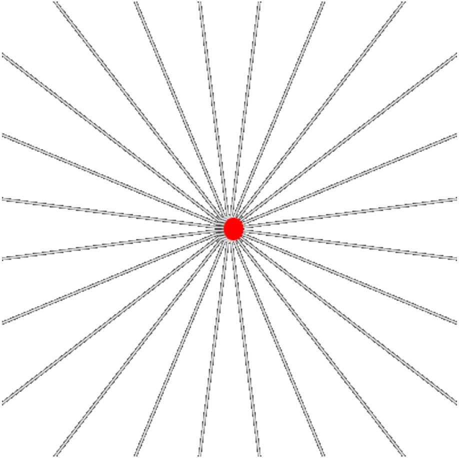
We saw that, by locality, a nudged charge sends out a “signal” which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

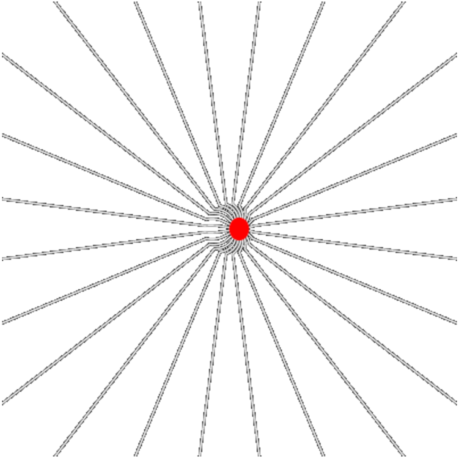
We saw that, by locality, a nudged charge sends out a “signal” which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

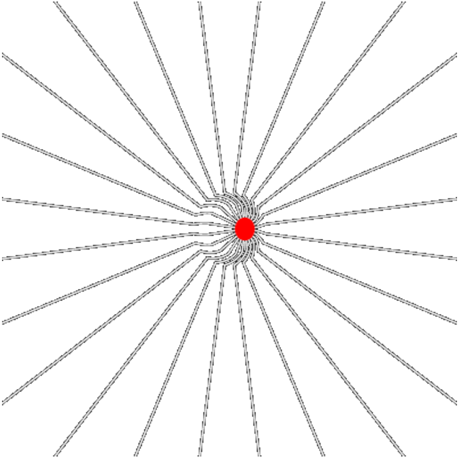
We saw that, by locality, a nudged charge sends out a "signal" which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

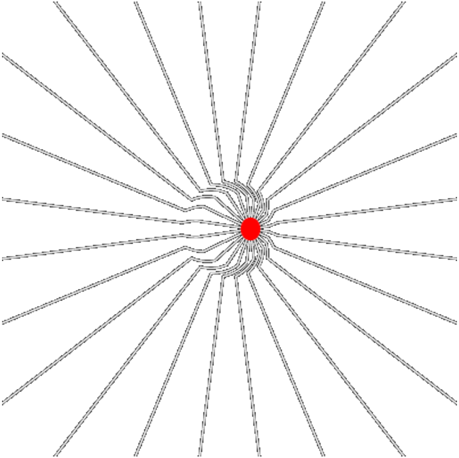
We saw that, by locality, a nudged charge sends out a “signal” which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

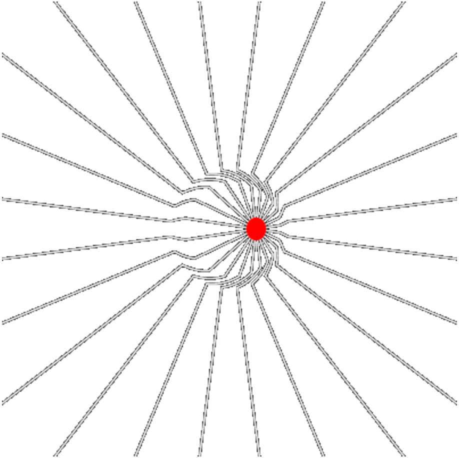
We saw that, by locality, a nudged charge sends out a "signal" which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

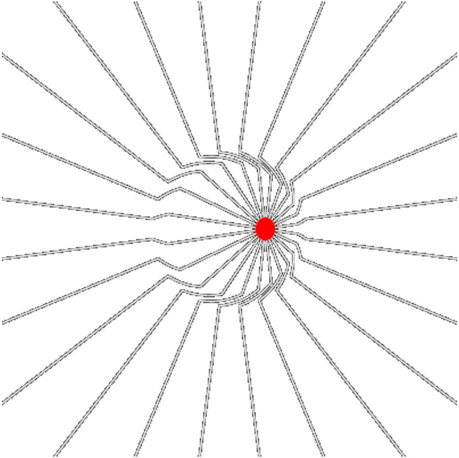
We saw that, by locality, a nudged charge sends out a "signal" which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

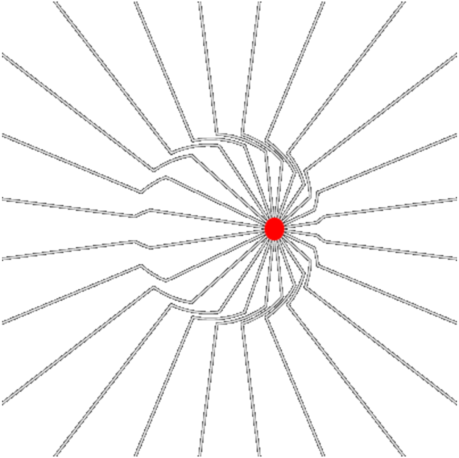
We saw that, by locality, a nudged charge sends out a "signal" which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

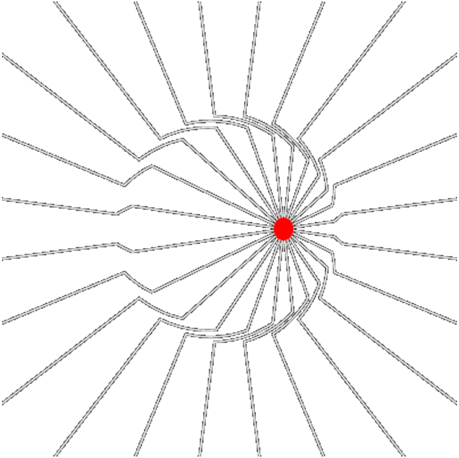
We saw that, by locality, a nudged charge sends out a “signal” which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

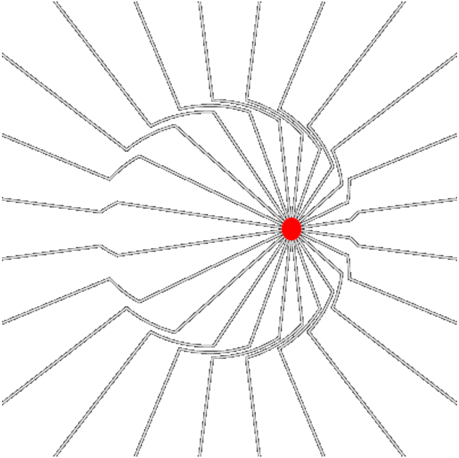
We saw that, by locality, a nudged charge sends out a "signal" which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

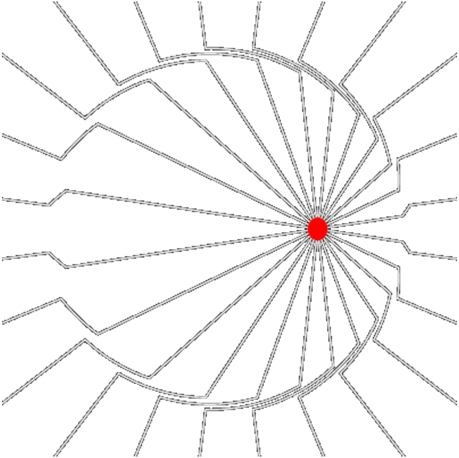
We saw that, by locality, a nudged charge sends out a "signal" which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

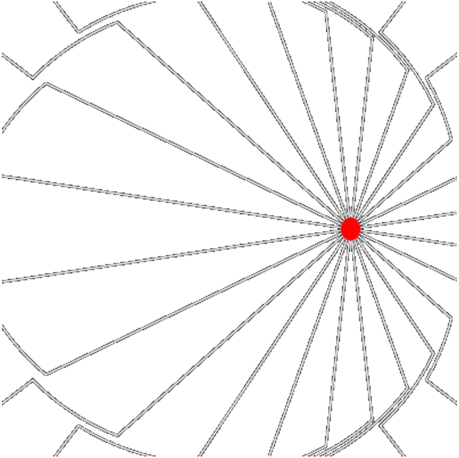
We saw that, by locality, a nudged charge sends out a "signal" which updates the electric field to point to its new location.



What is the speed of this signal?

A propagation speed.

We saw that, by locality, a nudged charge sends out a "signal" which updates the electric field to point to its new location.



What is the speed of this signal?

Two fundamental constants.

First, let's try to guess the speed. The laws of electromagnetism are called *Maxwell's equations*, which we will meet shortly.

The equations contain two special numbers, which describe the strength of electric fields and *magnetic fields*.

$$\text{Electric constant:} \qquad \epsilon_0 = 8.85 \times 10^{-12} \, \frac{\mathrm{s}^2 \, \mathrm{C}^2}{\mathrm{m}^3 \, \mathrm{kg}} \,,$$

$$\text{Magnetic constant:} \qquad \mu_0 = 1.26 \times 10^{-6} \, \frac{\mathrm{kg} \, \mathrm{m}}{\mathrm{C}^2} \,.$$

Here "m" means meters, "s" means seconds, "kg" means kilograms, and "C" means Coulombs (a unit measuring the amount of charge).

How can we use this to guess a speed? First, a simpler example.

Games with units.

If you forgot that distance = rate · time, you could figure out the distance you travel by going 60 mph for 2 hours using units:

$$60\,\frac{\text{miles}}{\cancel{\text{hour}}}\cdot 2\,\cancel{\text{hours}} = 120\,\text{miles}.$$

So if r has units of $\frac{\text{miles}}{\text{hour}}$ and t has units of hours, the equation

$$d = rt$$

is the *only* formula for a distance d in miles which has the correct units!

This strategy of playing games with units is called *dimensional analysis*.

Getting a speed.

We can use dimensional analysis to guess a formula for a speed that depends on μ_0, ϵ_0. If we multiply them,

$$\epsilon_0\mu_0 = \left(8.85 \times 10^{-12}\,\frac{\mathrm{s}^2\,\cancel{\mathrm{C}^2}}{\mathrm{m}^3\,\cancel{\mathrm{kg}}}\right) \cdot \left(1.26 \times 10^{-6}\,\frac{\cancel{\mathrm{kg}}\,\mathrm{m}}{\cancel{\mathrm{C}^2}}\right) = 1.1 \times 10^{-17}\,\frac{\mathrm{s}^2}{\mathrm{m}^2}.$$

We want something with units of $\frac{\mathrm{m}}{\mathrm{s}}$, so take one over the square root:

$$\frac{1}{\sqrt{\epsilon_0\mu_0}} = 3.0 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}.$$

You may recognize this as the speed of light!

Light from Maxwell.

Maxwell made this discovery in 1864, after which he commented:

> *The agreement of the results seems to show that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws.*

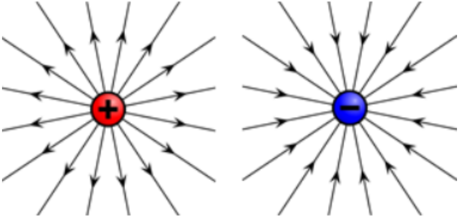
Punchline: visible light is *exactly* this traveling "stitch" that we got from moving a charge!

Maxwell did more than dimensional analysis: he actually wrote down wave solutions using his equations. Let's try to understand that next.

Act 2: a tour of Maxwell's equations.

Describing how arrows change.

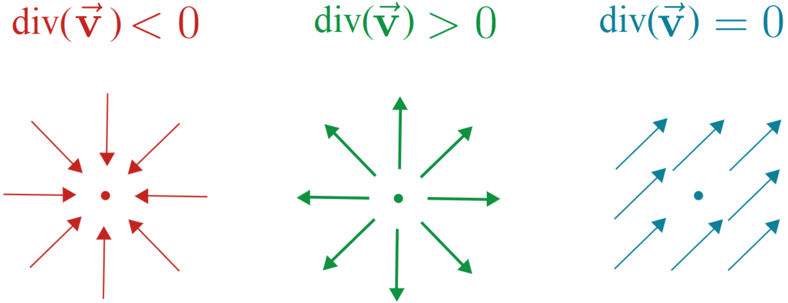
Maxwell's equations give a mathematical framework for describing how $\vec{E}$ and another field, the *magnetic field* $\vec{B}$, change in space and time.



Let's see how our qualitative description that "like charges repel and unlike charges attract" can be made quantitative. We already know that $\vec{E}$ points *away* from positive charges and *toward* negative charges.

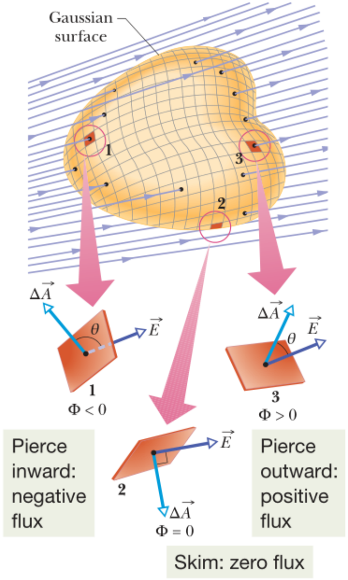
Divergence.

The mathematical tool for measuring how much arrows "point toward or away from" a location is the *divergence*.



If we imagine a pool of water where the arrows point in the direction the water is flowing, the divergence tells us whether a point is a *source* (water comes out) or a *sink* (water goes in). See here.

Defining divergence.



To calculate divergence at a point P:

1. Surround P by a surface.
2. Chop up the surface into patches.
3. Find flow of $\vec{E}$ through each patch.
4. Add up the flows from all patches.

The divergence is then

$$\mathrm{div}(\vec{E}) = \lim_{\text{volume}\to 0} \frac{\text{total flow}}{\text{volume}},$$

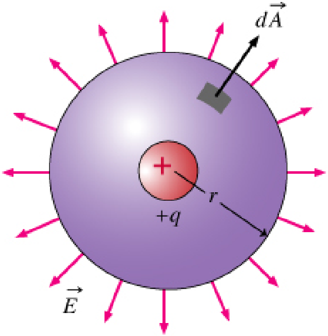
i.e. the ratio of the total flow from all patches, divided by the volume inside the surface, when the volume gets very small.

Gauss' law.

The first of Maxwell's equations is *Gauss' law*:

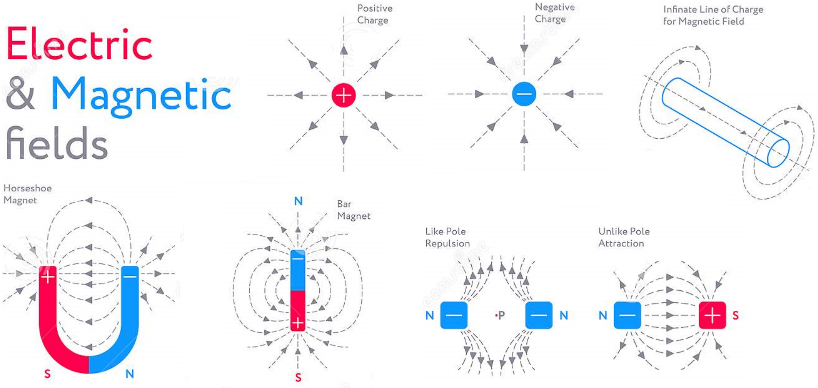
$$\mathrm{div}(\vec{E}) = \frac{\rho}{\epsilon_0}.$$

Here ρ is a function called the *charge density*. It is positive at the location of positive charges and negative at the location of negative charges.



Introducing the magnetic field.

The other Maxwell equations involve the *magnetic field*, written $\vec{B}$.



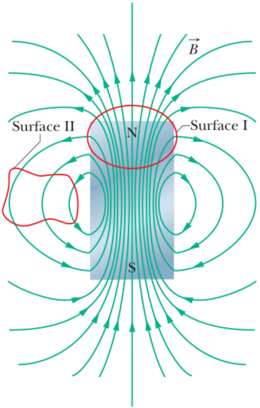
For now, think of $\vec{B}$ as being created by moving charges or magnets like the ones on your fridge.

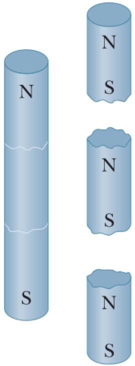
No magnetic charges.

The second Maxwell equation is

$$\mathrm{div}(\vec{B}) = 0.$$

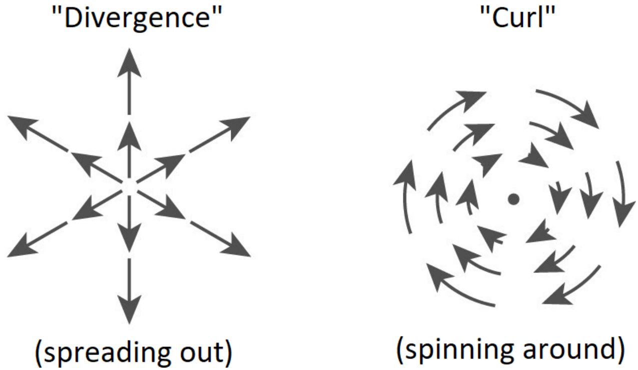
This says that there are no magnetic *charges*. In any small surface, the same amount of $\vec{B}$ "flows" in and out.





Curl.

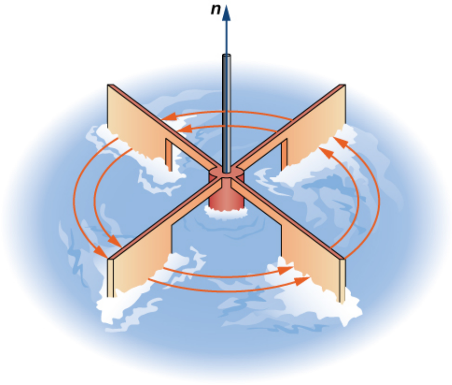
If $\vec{B}$ has no divergence, how else can we describe it? You may have noticed that $\vec{B}$ for a line of current seems to *spin* or *curl* around the current.



The mathematical tool for describing this behavior is called *curl*. It measures how much arrows spin around a point.

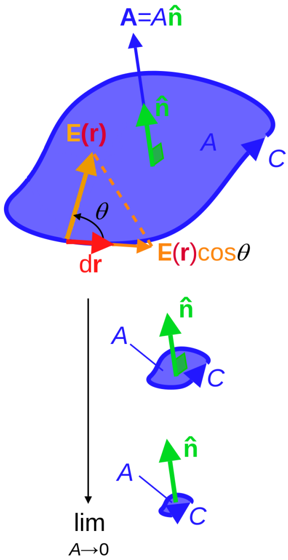
Curl visualized.

If we again imagine a pool of water where the arrows point in the direction of fluid flow, the curl tells us how much a small stick or paddlewheel would rotate if placed in the fluid.



See animation here.

Defining curl.



To calculate curl at a point P:

1. Surround P by a closed loop.
2. Chop up the loop into segments.
3. Project $\vec{E}$ onto each segment.
4. Add up the projections on all segments.

The curl* is then

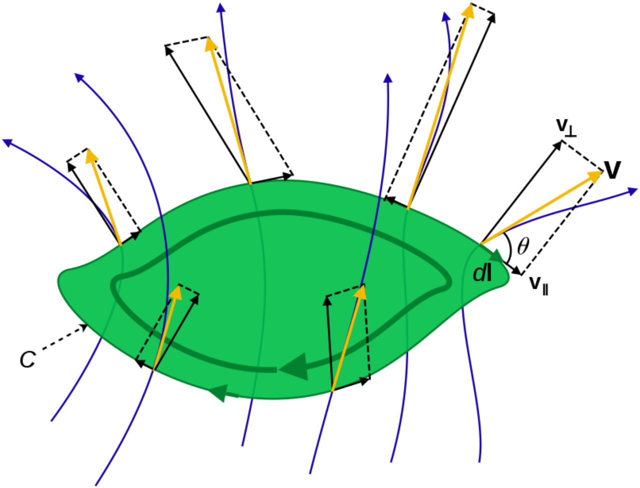
$$\mathrm{curl}(\vec{E}) \cdot \hat{n} = \lim_{\text{area}\to 0} \frac{\text{flow around loop}}{\text{area}},$$

i.e. the ratio of the total flow of $\vec{E}$ around the loop, divided by the area of the loop, when the area gets small.

*Technically, curl is a vector and this is just one component.

Curl measures circulation.

If $\vec{E}$ were the velocity of a fluid, then the projections onto the curve would measure how much the fluid is flowing in the direction of the curve.



Two more Maxwell equations.

The last equations tell us about curls:

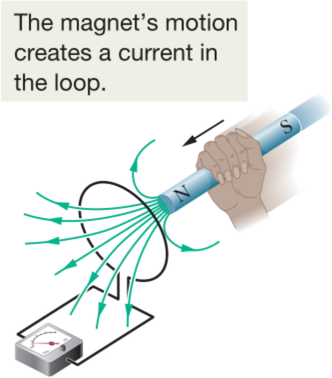
$$\begin{aligned} \mathrm{curl}(\vec{E}) &= -{}^{\mathrm{time}}_{\mathrm{change}}(\vec{B}), \\ \mathrm{curl}(\vec{B}) &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \, {}^{\mathrm{time}}_{\mathrm{change}}(\vec{E}). \end{aligned}$$

Here $\vec{J}$ is the current, which tells you where charges are moving.

Interpretation: a changing magnetic field produces an electric field, and a changing electric field produces a magnetic field!

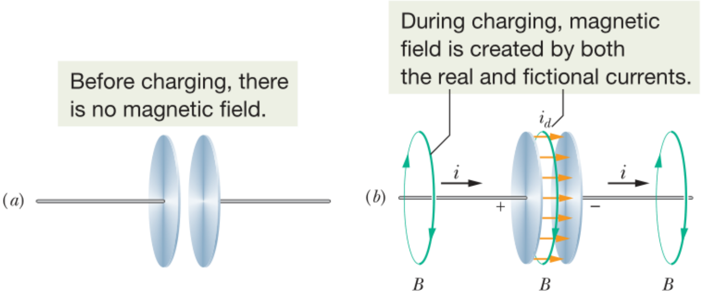
Induced $\vec{E}$.

Moving a magnet means $\vec{B}$ changes in time, which induces an electric field.



Induced $\vec{B}$.

Piling up charges on two plates means $\vec{E}$ changes in time, which induces a magnetic field.



Finally, the wave equation.

Here is a brief sketch of what Maxwell did (don't worry about details). In empty space, $\rho = 0$ and $\vec{J} = 0$. Combine Maxwell's equations to find

$$\mathrm{curl}(\mathrm{curl}(\vec{E})) = -\mu_0 \epsilon_0 \, {}^{\text{time}}_{\text{change}} \left({}^{\text{time}}_{\text{change}} (\vec{E}) \right).$$

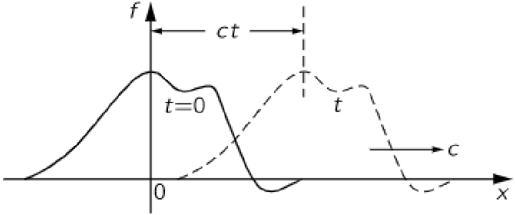
There is a theorem in math that, if $\mathrm{div}(\vec{E}) = 0$, then $-\mathrm{curl}(\mathrm{curl}(\vec{E}))$ is a special operator called the *Laplacian*. So

$$\mathrm{lap}(\vec{E}) = \mu_0 \epsilon_0 \, {}^{\text{time}}_{\text{change}} \left({}^{\text{time}}_{\text{change}} (\vec{E}) \right).$$

This is a famous relation called the *wave equation*!

Wave equation intuition.

The wave equation is simpler in one dimension, where its solution is any "moving graph" that travels in one direction at constant speed.



The function $f(x - ct)$ represents a constant "shape" that travels toward positive x with the speed c.

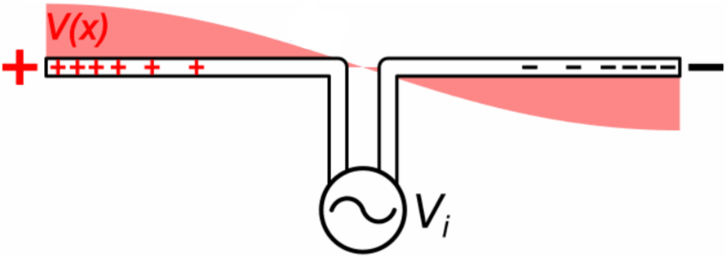
The constant speed in our wave equation is $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$. This gives a rigorous proof of the dimensional analysis claim we made in Act 1.

Interlude: dipole antennas.

What about real antennas?

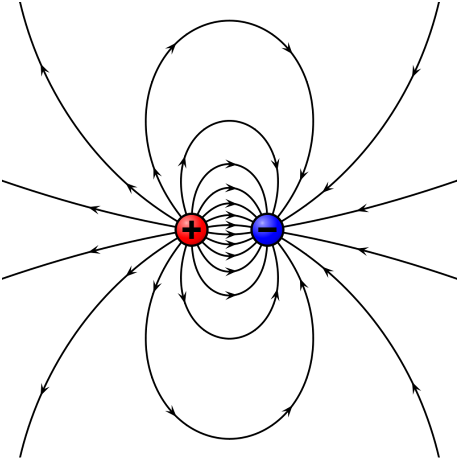
So nudging a lone electron generates radiation. But in real materials, we don't have lone electrons sitting around – they are bound to nuclei, like the copper in wires.

How do we nudge the electrons in a real antenna? First, we can apply an electric field that makes the electrons want to move to one end:



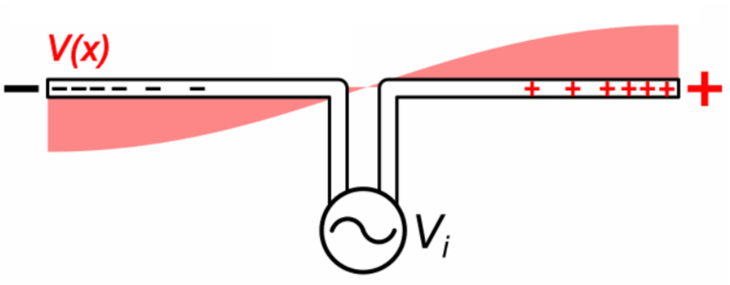
A dipole.

The resulting object – with positive charges bunched up on one end, and negative charges bunched up at the other end – is an *electric dipole*.



Reversing the dipole.

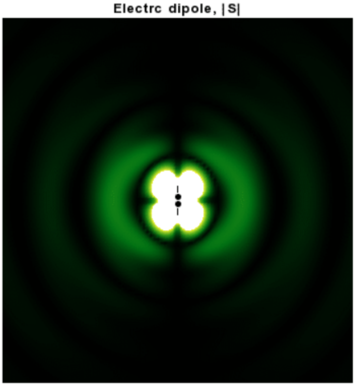
We need charges to move in order to generate radiation. So now we *reverse* the direction of the electric field. The electrons now get nudged to the opposite end of the antenna.



This doesn't happen all at once: the antenna smoothly transitions from one polarization to the other. Animation here.

Many small nudges $\longrightarrow$ signal.

As the electrons are nudged back and forth, a smooth signal moves outward to continually "update" the electric field to the current position of the charges.

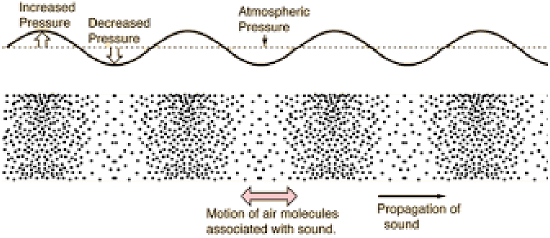


The signal carries energy, indicated by the brightness of the color plot above. Animations here and here.

Act 3: Einstein and the weird constancy of c.

What is waving?

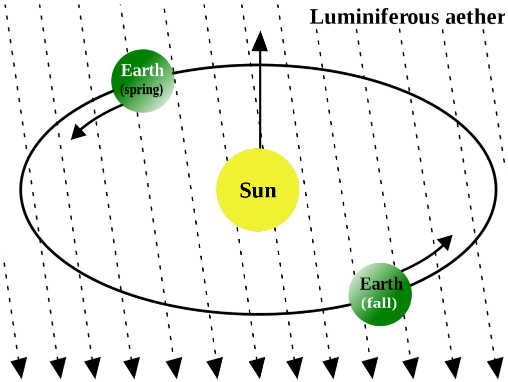
Light is an electromagnetic wave – but a wave in *what*? For other waves, like sound, there is a physical object (like air molecules) that is “waving”.



When we say “the speed of sound,” we mean the speed of this wave *relative to the air molecules*. What is “the speed of light” relative to?

Luminferous aether.

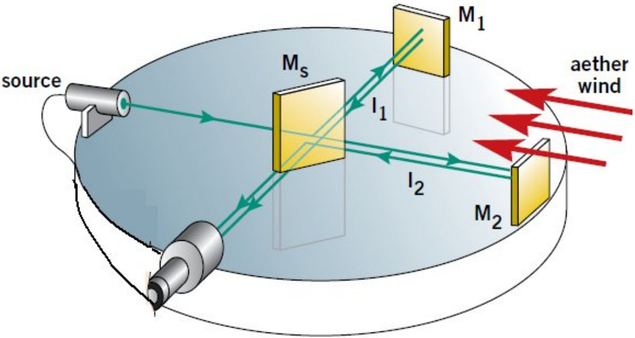
In the 1800s, it was believed that there was an invisible substance called "aether" that filled all of the universe, and light was a wave in the aether.



If this were true, the earth would be moving in different directions relative to the aether at different times of year.

But there is no aether.

However, in 1887, an experiment by Michelson and Morley showed that there is no aether.



This means that the "speed of light" is *not* measured relative to some substance, like air. The speed of light is a constant for all observers!

Two principles.

As with locality, it is extremely powerful to begin with basic principles and see where they take you. Einstein's 1905 paper assumed:

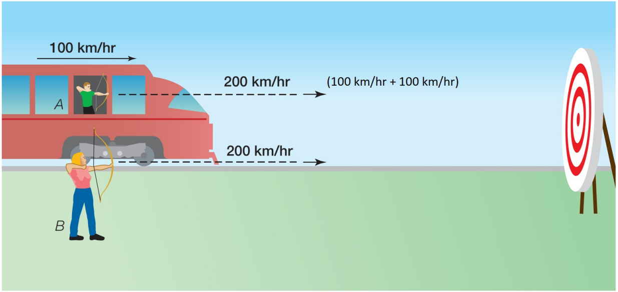
1. The speed of light is a constant for all observers.
2. The laws of physics should not depend on the physicist – the laws don't change if I am moving at some speed v.

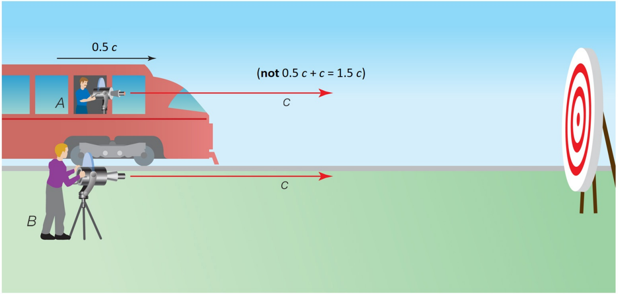
Assumption (1) is very strange! It means that speeds do not add in the way we usually expect.

If I am on a train going $100 \frac{\text{km}}{\text{hr}}$, and I shoot an arrow going $100 \frac{\text{km}}{\text{hr}}$, someone on the ground sees the arrow going $200 \frac{\text{km}}{\text{hr}}$. Speeds add.

But if I am on a train going at half the speed of light, and I turn on a laser with light going at c, someone on the ground **also** sees the laser light moving at c.

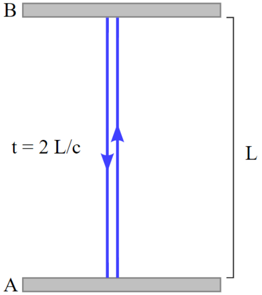
Arrows versus lasers.





Light bouncing between mirrors.

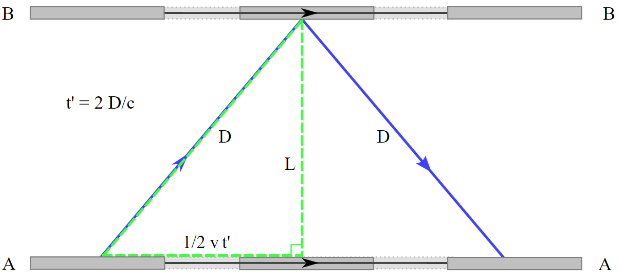
Suppose we have two mirrors, A and B, separated by a distance L. We turn on a light at the bottom mirror and wait for it to come back.



The light travels distance L upward, reflects off mirror B, and travels another distance L downward. Using $d = rt$, where the rate r is the speed of light c, we find that this takes a time $t = \frac{2L}{c}$.

Mirrors on a train.

Now we put the mirrors on a train moving to the right at speed v. We watch the light bounce between the mirrors from outside the train.



c is the same, so the total time it takes is now $t' = \frac{2D}{c}$, where D is the length of the hypotenuse in the picture. By Pythagoras,

$$D^2 = L^2 + \left(\frac{1}{2} v t' \right)^2 .$$

Some algebra.

We had $t' = \frac{2D}{c}$ and $t = \frac{2L}{c}$. Re-arranging gives $D = \frac{ct'}{2}$ and $L = \frac{ct}{2}$. Plugging both into Pythagoras gives

$$D^2 = L^2 + \left(\frac{1}{2}vt'\right)^2 \longrightarrow \left(\frac{ct'}{2}\right)^2 = \left(\frac{ct}{2}\right)^2 + \left(\frac{1}{2}vt'\right)^2.$$

Multiply through by 4 and move everything with t'^2 to the left:

$$c^2 t'^2 - v^2 t'^2 = c^2 t^2.$$

Dividing by c^2 and factoring, we find

$$t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Time dilation!

Punchline: A moving clock runs slow!

If you look at a clock moving with speed v relative to you, then you see that clock running slow by a factor of $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$:

$$t_{\text{moving}} = \gamma t_{\text{proper}}.$$

This prediction has been thoroughly tested, perhaps most famously in the Hafele-Keating experiment, where people put very accurate atomic clocks on airplanes.

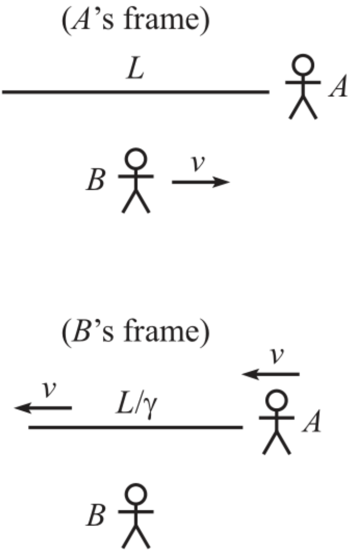
Length contraction.

Person A stands still next to a stick of length L while person B runs by at speed v. Person A's clock says that it takes a time $t_A = \frac{L}{v}$ for person B to pass the stick

Person A sees person B's clock run slow. So to person B, it takes a *shorter* time $t_B = \frac{t_A}{\gamma} = t_A\sqrt{1 - \frac{v^2}{c^2}}$ for the stick to pass by.

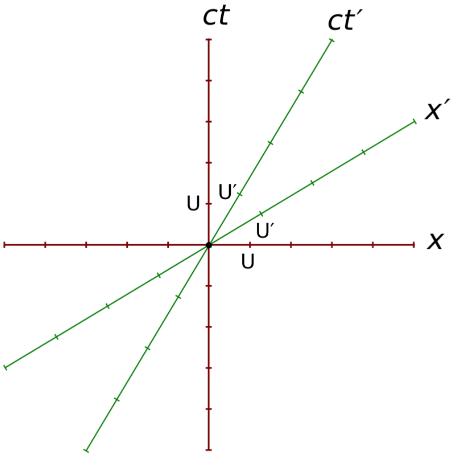
Person B uses $d = rt$ to calculate the length of the stick and finds:

$$L_B = v \cdot t_B = \frac{v t_A}{\gamma} = L\sqrt{1 - \frac{v^2}{c^2}}.$$



Spacetime.

Maxwell and Einstein are teaching us that space and time do not, on their own, have absolute meaning. Only a combination of the two called *spacetime* is meaningful.



Epilogue: magnetism and principles of theoretical physics.

Principles simplify science.

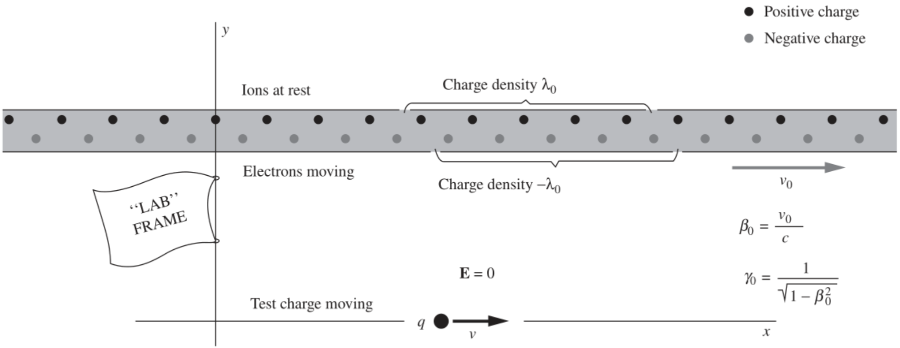
Here is the big picture takeaway of the talk:

> **Punchline**: starting with simple physical principles, like locality, and exploring their logical consequences can lead to remarkable scientific discoveries.

Locality told us that a nudged charge must radiate. Constancy of c told us that lengths contract and times dilate. And for one final example, let's see how length contraction tells us there must be a magnetic force.

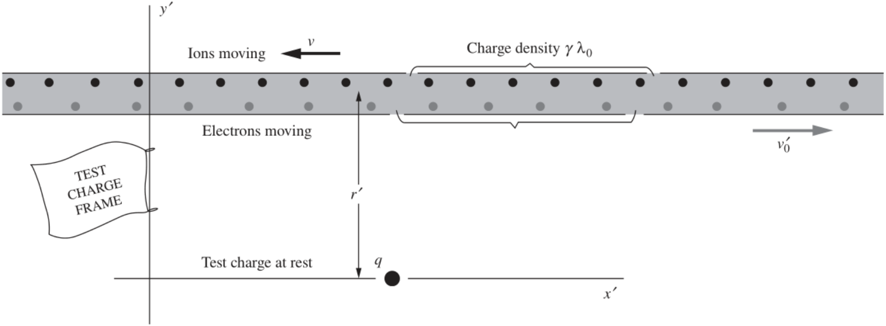
Magnetism from relativity.

Suppose I have a copper wire with a current, which means the electrons are moving. A positive test charge moves outside the wire.

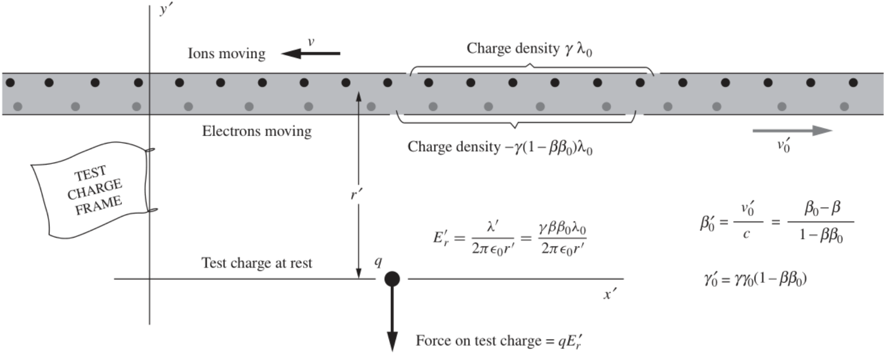


Relativity at work.

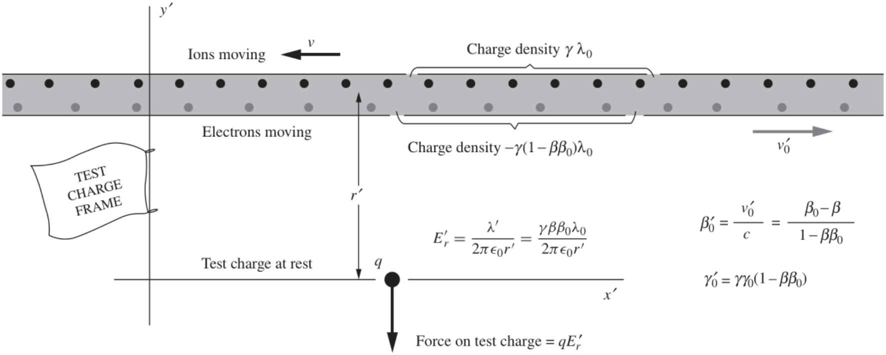
Go to the positive charge's frame. By length contraction, the distance between copper ions is smaller. So there is a *larger* charge density of positive ions.



The electrons were originally moving in the same direction as the positive charge. So to the positive charge, the electrons appear to be moving *slower*. They have a *smaller* length contraction than the copper ions.



Because the electrons have a *smaller* charge density, the wire appears positively charged. The positive charge outside the wire is *repelled*.



Magnetism!

If the positive charge is repelled in one frame, it is repelled in all frames. But in the "lab frame," there is no $\vec{E}$ field because the wire is neutral.

This means that there must be a *second* force, due to a different field, that acts on moving charges. This is the *magnetic field* $\vec{B}$.

The magnetic force causes a positive charge moving near a wire to be repelled from the wire, if electrons in the wire are moving in the same direction as the positive charge.

Lorentz invariance is powerful.

Compatibility with relativity – which we call "Lorentz invariance" – requires not just an electric force, but a magnetic force.

If you go on in physics, you will learn other principles which make this fact seem obvious. A modern physicist would immediately tell you that $\vec{E}$ and $\vec{B}$ must go inside a matrix called the field strength tensor:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}.$$

I hope some day you will find out why!

The end.

Thank you for your attention! I had a lot of fun preparing this talk.

If you have questions about

- this talk,
- physics,
- applying to competitive colleges,
- being a student at MIT,
- doing a PhD,
- or anything else,

please email cferko@alum.mit.edu.