Advanced Topics in Combinatorics	**Summer 2018**

Lecture 2: Partitions

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2.1 Introduction

A *partition* of an integer n into k *parts* is a nonincreasing tuple of positive integers $(a_1, a_2, \dots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$. We will denote the number of partitions of n into at most k parts as $p_k(n)$, and the total number of partitions of n into any number of parts as $p(n)$. For example, $p(5) = 7$, with the associated partitions being $(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)$, and $p_3(5) = 5$ since all but the last two of these have at most 3 parts.

Now, let $p^k(n)$ be the number of partitions of n into parts of size at most k. For example, $p^3(5) = 5$, corresponding to the partitions $(3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)$. Notice curiously that $p^3(5) = p_3(5)$. Checking this for other values of n and k, we see similarly that $p^2(5) = p_2(5) = 3$, $p^2(4) = p_2(4) = 3$. This leads us to the following proposition

Proposition 2.1. $p^k(n) = p_k(n)$.

Proof: To prove this, consider the Ferrer diagram representation of any partition, pictured below for the partition $(5,3,3,2,1,1)$, where the rows represent the distinct parts of the partitions, the number of dots in each row the size of the part.

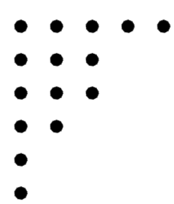

Figure 2.1: The Ferrer diagram for the partition $(5,4,3,2,1,1)$.

Any such partition has a *conjugate partition*, in which the rows and columns are transposed. For example, the two Ferrer diagrams below are conjugate.

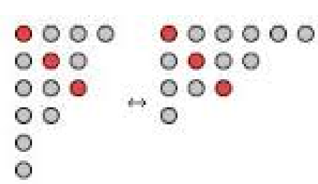

Figure 2.2: Conjugate Ferrer diagrams, (4, 3, 3, 2, 1, 1) and (6, 4, 3, 1).

Any partition P clearly has a corresponding conjugate partition, which we'll denote by P^T. In particular, now, note that any partition P_k with parts of size at most k has the property that P_k^T has at most k parts (this can be easily seen since we are just flipping the rows and columns of our Ferrer diagram). In particular, this serves as a bijection between partitions with at most k parts and those with parts of size at most k, since this operation is reversible and one-to-one, and thus the sets are equal. In other words, as desired, $p^k(n) = p_k(n)$. ■

Now, let us consider the generating function of $p^k(n)$. It might seem intimidating at first, but notice that we can entirely describe a partition by the number of 1's, 2's, 3's, etc.; putting this in the language of generating functions, we can write

$$\sum_{n=0}^{\infty} p^k(n)x^n = (1 + x + x^2 + \cdots)(1 + x^{2\cdot 1} + x^{2\cdot 2} + \cdots)\cdots(1 + x^k + x^{2k} + \cdots),$$

since choosing a term from the multiplicand corresponding i corresponds to choosing a particular number of i's in your partition, so each part in the corresponding partition is of size at most k. For example, if we let $k = 2$, we can write out the first few terms of $(1+x+x^2+\cdots)(1+x^{2\cdot 1}+x^{2\cdot 2}+\cdots) = 1+x+2x^2+2x^3+3x^4+3x^5+\cdots = p_2(0) + p_2(1)x + p_2(2)x^2 + p_2(3)x^3 + \cdots$, as desired. Since we know from the sum of infinite geometric series that $1 + x^k + x^{2k} + \cdots = \frac{1}{1-x^k}$, we then get that

$$\sum_{n=0}^{\infty} p^k(n)x^n = \prod_{i=1}^{k} \frac{1}{1 - x^i}.$$

Taking this over all partitions gives similarly that

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}.$$

Isn't this nice? We will now put this generating function (and the ideas behind it) to good use, first by proving the following.

Proposition 2.2. *Let $p_o(n)$ be the number of partitions of n into an odd number of parts, and $p_u(n)$ be the number of partitions of n into unequal parts. Then we have that $p_o(n) = p_u(n)$.*

Proof: First, note that, by Proposition 3.1, $p_o(n)$ is the same as the number of partitions of n into parts of odd size. From this, by the same logic as for the previous generating function, $p_o(n) = \displaystyle\prod_{i=0}^{\infty} \frac{1}{(1 - x^{2i+1})}$. Now, the generating function for $p_u(n) = (1 + x)(1 + x^2)(1 + x^3)\cdots$, because we can only include zero or one part of each size, if all the parts are to be unequal. Now, note that $1 + x^k$ can also be written as $\frac{1-x^{2k}}{1-x^k}$, so $p_u(n) = \prod_{k=1}^{\infty} \frac{1-x^{2k}}{1-x^k}$. Now, note that all of the terms in the numerator of this product also appear in the denominator at some point, so the entire numerator cancels, and the terms that remain in the numerator are those of the form $1 - x^k$ for odd j. In other words, $p_u(n) = \prod_{j=0}^{\infty} \frac{1}{1-x^{2j+1}} = p_o(n)$, as desired, so we are done. ■

2.2 More Partition Identities

In this section, we'll present some more well-known partition identities; in other words, facts about partitions that give us algebraic facts about generating functions. The first involves the so-called Durfee square; I'll first present the intuition behind this, and then give a formal proof for your reading pleasure in the lecture notes. A Durfee square, for a given partition represented by a Ferrer diagram as above, is the largest size

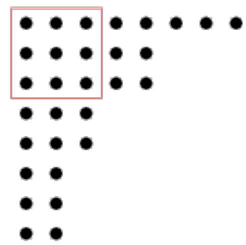

Figure 2.3: The partition $(8, 5, 5, 3, 3, 2, 2, 2)$, with its associated 3×3 Durfee square

of a square that could fit in the upper left corner of that diagram; for example, in the diagram below, the Durfee square is of size 3, as shown in Figure 3.3.

Intuitively, all such partitions have such a unique square, and thus if we sum over all partitions, using a generating function based on this Durfee square, we should get the total number of partitions. This gets us the following result.

Problem 2.2.1. *Show that*

$$\sum_{k=0}^{\infty} \frac{x^{k^2}}{((1-x)(1-x^2)\cdots(1-x^k))^2} = \prod_{n=1}^{\infty} \frac{1}{1-x^n}.$$

Solution: Consider any partition $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n with $\lambda_1 \geq \cdots \geq \lambda_n$, and consider the largest $k \in \mathbb{N}$ such that $\lambda_k \geq k$. This is clearly unique for a given partition. Consider all partitions for a fixed such k; then, since $\lambda_i \geq k$ for $1 \leq i \leq k$, k^2 of the partition is fixed. The rest of the partition consists of two components; first, we know that each of the λ_i for $1 \leq i \leq k$ can have additional components, so we can append to these λ_i any partition with less than or equal to k parts. Similarly, for $i \geq k$, all such λ_i must be less than k, since otherwise k would not be the largest integer for which $\lambda_k \geq k$, and thus we can append any partition with all parts less than or equal to k. This gives us that the total number of partitions of n for a fixed such k is given by the coefficient of n

$$x^{k^2} \cdot \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} \cdot \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} = \frac{x^{k^2}}{((1-x)(1-x^2)\cdots(1-x^k))^2},$$

where the first part of the product derives from the fixed k^2 and the second two derive from the partition identities derived early for $p_k(n)$ and $p^k(n)$. Since every such partition clearly has a unique such k, if we sum over all k, we get all the partitions of n as the coefficient of x^n, and thus we get that

$$\sum_{k=0}^{\infty} \frac{x^{k^2}}{((1-x)(1-x^2)\cdots(1-x^k))^2} = \sum p(n)x^n.$$

But by our generating function for partitions, we have that $\sum p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$, and thus indeed

$$\sum_{k=0}^{\infty} \frac{x^{k^2}}{((1-x)(1-x^2)\cdots(1-x^k))^2} = \prod_{n=1}^{\infty} \frac{1}{1-x^n}.$$

■

Another, fascinating result involves showing that the number of partitions with distinct, odd parts is equal to the number of self-conjugate partitions, or partitions which are conjugates of themselves.

Proposition 2.3. *Let $p_{od}(n)$ be the number of partitions of n into distinct, odd parts, and $p_{sc}(n)$ be the number of self-conjugate partitions of n. Then $p_{od}(n) = p_{sc}(n)$.*

We'll prove this result below; before we do, however, note that interestingly enough, this result is helpful in establishing the parity of $p(n)$ for a general n. In particular, the following holds, given that the above is true:

Proposition 2.4. *Let $p(n)$ be the number of partitions of n, and $r(n)$ be the number of partitions of n into distinct odd numbers. Then $p(n) \equiv r(n) \bmod 2$.*

Proof: Let A be the set of non-self conjugate partitions of n, and B the set of self-conjugate partitions of n. Clearly $|A|+|B|= p(n)$. Since conjugation is an involution (in other words, applying it twice yields the other partition), A can be split into distinct pairs of conjugates (a_1, a_2). Thus $|A|$ is even, so $p(n)$ has the same parity as $|B|$, which, by Proposition 3.3, implies that $p(n)$ has the same parity as the number of partitions of n into distinct odd numbers, $r(n)$, as desired. ■

We'll now prove Proposition 3.3 in a rigorous fashion; in lecture, we will only cover the intuition behind this, but for the reader's interest, a rigorous proof is given below.

Proof: To prove this result, we will describe a bijection between the set of partitions into distinct, odd parts and the set of self-conjugate partitions. In particular, consider a partition into distinct odd numbers $o_1, o_2, \ldots, o_j$, $o_1 \geq o_2 \geq o_3 \ldots \geq o_j$, where each odd number is of the form $2k_i - 1$ for nonnegative integers k_i, and consider the following process (described loosely in algorithmic fashion) to create a new partition: for each o_i, let $\lambda_i = \lambda_i + k_i$ and $\lambda_{i+j} = \lambda_{i+j} + 1$ for $1 \leq j \leq k_i$. In this manner, we both increase a row by k_i and also add a single element to k_i rows. We claim that the resultant partition is self-conjugate; we will prove this by induction, with our invariants at the ith iteration being that the partition is self-conjugate and that the ith column of the associated diagram contains at least $i + k_i - 1$ boxes. The base case is before any iteration of the process occurs; clearly, the empty partition is self-conjugate and has an 0th column of size $0 + 1 - 1 = 0$, so this is satisfied. Then, assume we have iterated this process for i of the odd numbers and we are adding the $(i + 1)$st. This process increases the $(i + 1)$st row by k_i and also adds k_i elements to the $(i+1)$st column; to see this latter fact, note that, because $o_{i+1} < o_i, k_{i+1} \leq k_i - 1$, $i+1+k_{i+1}-1 \leq i+k_i-1$, so this process is adding 1 to rows which already have a box in the ith column, which means that it is adding them to the $(i + 1)$st column. Since it adds the same amount of elements to the $(i + 1)st$ column and row, this column/row is invariant under transposition; since the rest of the partition is also self-conjugate by our inductive step, the resultant total partition is self-conjugate after the $(i + 1)$st step, and we are done.

Intuitively, this process is equivalent to creating "symmetric hooks" out of the odd numbers and nesting these hooks. It is clearly invertible, since we simply "unravel" each of the hooks to create a distinct odd partition, and thus it is indeed injective. This can be seen in the diagram below:

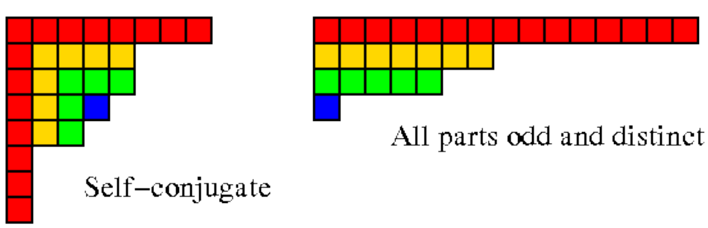

Figure 2.4: *The process for converting a self-conjugate partition into a partition with distinct, odd parts, which consists of "unraveling" the hooks in red, yellow, green, and blue, and laying them out horizontally. As seen in the diagram, the resultant partition satisfies the desired properties, and this is a reversible process.*

To be more rigorous, from a self-conjugate partition, since $l(i)$ the height of the ith column is equal to λ_i (since it is invariant under transposition), we can create a distinct odd part partition by letting $o_i = 2(\lambda_i - i) + 1$ until this process creates a negative number. This is exactly the inverse of the process described above. Since $\lambda_1 \geq \lambda_2 \geq \cdots$, we have that $o_i = 2(\lambda_i - i) + 1 > 2(\lambda_i - i - 1) + 1 \geq 2(\lambda_{i+1} - i - 1) + 1 = o_{i+1}$, so the odd numbers generated by this process are distinct. This inverse process gives both the injectivity of our first map and, because, given a self-conjugate partition, we can perform this inverse process to generate a distinct odd partition that maps to it under the original process, and thus the original map is bijective, and we are done. ■

2.3 Euler's Theorem

Our next topic of interest is Euler's "Pentagonal Numbers" Theorem, which concerns the number of partitions into an even and odd number of unequal parts. In particular, let $p_e(n)$ and $p_o(n)$ denote these numbers. Observe that the following table gives the values of these functions for $n = 3$ to 10

n	$p_e(n)$	$p_o(n)$
3	1	1
4	1	1
5	2	1
6	2	2
7	3	2
8	3	3
9	4	4
10	5	5

As you can see, they are almost always equal. Euler's Theorem, which we'll prove shortly, formalizes this in an interesting way.

Theorem 2.5. *For all n, we have that*

$$p_e(n) - p_o(n) = \begin{cases} (-1)^k & \text{if } n = \frac{3k^2 \pm k}{2} \\ 0 & \text{otherwise} \end{cases}.$$

Note that this number, $\frac{3k^2-k}{2}$, is called pentagonal because it is exactly the number of dots in a pattern of dots tracing regular pentagons with sides of length up to k. We could devote a whole lecture to properties of triangular, square, pentagonal, and hexagonal numbers, but suffice to say that they appear in many, many places.

Now, we prove Theorem 3.5. We will not prove it exceptionally rigorously, but rather simply describe the two transformation that lead to its truth.

Proof. To prove this theorem, we will define two transformations on partitions with unequal parts, which will serve as bijections between these sets of partitions.

Consider the partition in question, with its parts $a_1, a_2, \ldots, a_j$. Let $b = a_j$, the base, and s be the slope of the partition, or the number of dots in the 45 degree line through the upper-right dot of the Ferrers diagram. These two concepts are depicted in Figure 3.5.

Now, our goal is to remove or add a row, while keeping the fact that all of the rows have an unequal number of parts. If $b \leq s$, then we can delete the base, and add an additional dot to each of the rows above

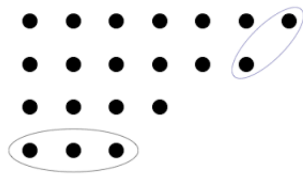

Figure 2.5: The base (black) and slope (blue) for the partition $(7, 6, 5, 3)$, of size 3 and 2 respectively.

it, thus creating a partition of unequal parts of the opposite parity. This only fails if $b = s$ and the base and slope overlap, which gives that $n = \frac{3b^2-b}{2}$, one of our exceptions, as desired. The other case, if $b > s$, then we do the opposite, deleting the slope and adding it as a base; this only fails if $b = s + 1$ and the slope overlapped with the base, in which case the new base and the row above it have the same number of parts. This, however, with some computation can be shown to imply that $n = \frac{3(b-1)^2+(b-1)}{2}$, another one of our exceptions. An example of a success of this second transformation is shown in Figure 3.6; an example of the first can be seen if we simply reverse the step in Figure 3.6.

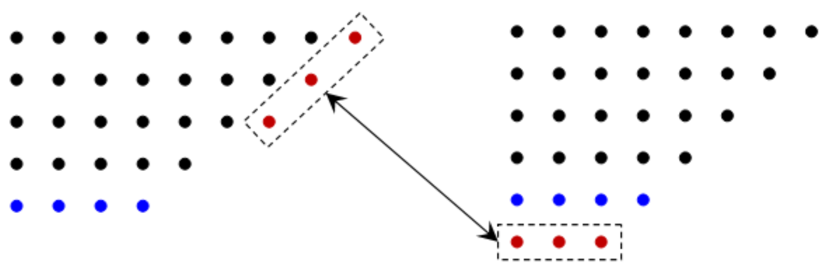

Figure 2.6: The second transformation described, depicted between the partitions $(9, 8, 7, 5, 4)$ and $(8, 7, 6, 5, 4, 3)$.

From these two transformations, we have a bijection between $p_e(n)$ and $p_o(n)$ in the non-exceptional cases, implying their equality and thus their zero difference. In the exceptional case, since there is only one possible setup that causes an exception, their difference must be ± 1; the parity can be found by examining the above transformations a little more closely, which we will not concern ourselves with, since it was covered in lecture. ☐

Now, note that the generating function for $p_e(n) - p_o(n)$ is also evidently $\prod_{i=1}^{\infty}(1 - x^i)$, since any partition into k unequal parts contributes $(-1)^k$ to the coefficient of x^n, as desired. Noting that $\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$, if we multiply this by the generating function given above, we have that

$$\left(\sum_{n=0}^{\infty} p(n)x^n\right)\left(\sum_{k=1}^{\infty}(-1)^k(x^{(3k^2-k)/2} + x^{(3k^2+k)/2})\right) = 1,$$

which, looking at the coefficient of x^n for $n \geq 1$, implies that $p(n) = \sum_{k=1}^{\infty}(-1)^{k+1}\left(p\left(n - \frac{3k^2 \pm k}{2}\right)\right)$. This is an astonishing result! This gives us a recurrence for $p(n)$ that allows to efficiently calculate the partition numbers with just approximately $2\sqrt{2n/3}$ terms, which is a huge improvement – allowing us to calculate $p(1000)$ with just 50 previous values, for example.

2.4 Optional: bounds on $p(n)$

A really interesting area of interest is an approximate bound on $p(n)$. The following result and proof is due to Van Lint, although we'll discuss briefly an improvement of it afterwards.

Proposition 2.6. *For $n > 2$, we have $p(n) < \frac{\pi}{\sqrt{6(n-1)}} e^{\pi\sqrt{2n/3}}$.*

Proof. First, consider the generating function for $p(n)$, $P(x) = \sum_{k=1}^{\infty} \frac{1}{1-x^k}$. We are going to find an upper bound for $P(x)$, and then pick an x such that the dominating term in the generating function is $p(n)x^n$, thus giving us a bound. To get the upper bound, we take logs, getting that

$$\log(P(x)) = \prod_{k=1}^{\infty} \log\left(\frac{1}{1-x^k}\right).$$

Now, note that $\log\left(\frac{1}{1-x^k}\right) = -\log(1-x^k)$. Expanding out the power series for log, this gives that $\log(P(x)) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{x^{kj}}{j} = \sum_{j=1}^{\infty} \frac{1}{j}\frac{x^j}{1-x^j}$, if we exchange and recollapse terms. From here, noticing that $1-x^j = (1-x)(1+x+x^2+\cdots x^{j-1}) > j(1-x)x^{j-1}$, we get that $\frac{x^j}{1-x^j} < \frac{x^j}{j(1-x)x^{j-1}} = \frac{x}{j(1-x)}$. This in turn gives that, plugging into our expression for $\log(P(x))$, that $\log(P(x)) < \frac{x}{1-x}\sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{x}{1-x}\cdot\frac{\pi^2}{6}$! That's where the π comes from!

Now, we want a lower bound on $P(x)$, which we clearly know is greater than a subset of itself generating function, or $\sum_{k=n}^{\infty} p(k)x^k$. Noting that $p(n)$ is increasing, this in turn is greater than $p(n)\sum_{k=n}^{\infty} x^k = \frac{p(n)x^n}{1-x} <$ $P(x)$, as we have shown. Taking logs of both sides and inserting our upper bound, we get that $\log(p(n)) <$ $\frac{x}{1-x}\cdot\frac{\pi^2}{6} - n\log(x) + \log(1-x)$. If we substitute in $u = \frac{1-t}{t}$ (which means that $t = \frac{1}{1+u}$) for x, as inspired by Van Lint, we get that our given expression implies that $\log(p(n)) < \frac{\pi^2}{6u} + n\log(1+u) + \log(\frac{u}{1+u}) =$ $\frac{\pi^2}{6u} + (n-1)\log(1+u) + \log(u) < \frac{\pi^2}{6u} + (n-1)u + \log(u)$, using the fact that $\log(1+u) < u$, since $e^u = 1 + u + \sum_{k=2}^{\infty}\frac{u^k}{k!}$.

Now, our result follows from plugging in the value of u that makes $(n-1)u = \frac{\pi^2}{6u}$, which we can solve to be $u = \frac{\pi}{6\sqrt{n-1}}$. □

Note that Hardy and Ramanujan slightly improved the constant in this result, showing the following.

Theorem 2.7 (Hardy, Ramanujan)**.** *Let $p(n)$ denote the number of partitions of n. Then $p(n) \asymp \frac{1}{4\sqrt{3}n}e^{\pi\sqrt{\frac{2}{3}n}} =$ $f(n)$, which implies that $\lim_{n\to\infty}\frac{p(n)}{f(n)} = 1$.*

2.5 The Genius of Ramanujan and Looking Forward

There's a lot of history behind partitions and the identities discovered about that, and if you read more about partitions in other papers online, you'll hear a lot about various mathematicians that contributed greatly to the field of study, with many more results than the few we could teach here. However, one above all, the unschooled Indian mathematician Ramanujan, in his collaboration with Hardy, dominated the field, and produced a plethora of results. In honor of him and the recent release of the Man Who Knew Infinity, a movie about his life that I'd recommend you all go see, I'll list some of them here, without proof; this is just a glimpse into the genius that was Ramanujan.

First, we have the Rogers-Ramanujan identities, which derived from Ramanujan's analysis of the Rogers-Ramanujan functions. They are the following, interpreted on partitions (they also have interpretations as hypergeometric functions, as do many of these results):

Theorem 2.8 (Rogers-Ramanujan's First Identity). *The number of partitions of n such that adjacent parts differ by at least 2 is the same as the number of partitions of n such that each part is either 1 or 4 mod 5.*

Theorem 2.9 (Rogers-Ramanujan's Second Identity). *The number of partitions of n such that adjacent parts differ by at least 2 and the smallest part is at least 2 is the same as the number of partitions of n such that each parts is either 2 or 3 mod 5.*

It turns out, in fact, that these can be generalized to the following beautiful result:

Theorem 2.10 (Gordon's partition theorem, 1961). *Let $A_{k,i}(n)$ denote the number of partitions of n into 0, i, or $-i$ mod $2k+1$, and $B_{k,i}(n)$ denote the number of partitions of n in which 1 appears at most $i-1$ times and the total number of consecutive numbers in the partition is at most $k-1$. Then $A_{k,i}(n) = B_{k,i}(n)$.*

Note that this implies the two identities of Rogers and Ramanujan in the cases $(k,i) = (2,2),(2,1)$, which were proven 40 years prior with much less development in the associated mathematics. It is quite extraordinary that they were able to find these results so early in the development of the field, especially with Ramanujan being very much isolated from the rest of mathematics. The proof of the Gordon partition theorem is done bijectively, like many of our proofs above – though we do not have space in this lecture to do it, know that it is not incredibly complicated, and can also be done solely analytically, if you try hard enough.

Ramanujan also proved the following congruences modulo primes regarding partitions, which have turned out to spawned two open conjectures.[1]

Proposition 2.11 (Ramanujan). *For any integer n, $p(5n+4) \equiv 0 \bmod 5$.*

Proposition 2.12 (Ramanujan). *For any integer n, $p(7n+5) \equiv 0 \bmod 7$.*

Proposition 2.13 (Ramanujan). *For any integer n, $p(11k+6) \equiv 0 \bmod 11$.*

These congruences were proved using what is called the *rank* of the partition, or the largest part minus the number of parts of the partition. It turns out, with some clever algebraic tricks, you can use this rank to divide any partition, in the above cases, into groups of size 5, 7, or 11, respectively, thereby showing the desired results.

One might be tempted to generalize these results to conjecture that $p(13n+7) \equiv 0 \bmod 13$, but, as it turns out, $p(7) = 15$. However, there do exist similar congruences for further primes:

[1] Note that much of my discussion here comes from a similar article on integer partitions, which can be found here: `http://cecas.clemson.edu/~jimlb/Teaching/Math573/Math573partitions1.pdf`

Proposition 2.14 (Atkin). *For any integer n, $p(17303n + 237) \equiv 0 \bmod 13$.*

Notice that $17303 = 11^3 \cdot 13$.

Proposition 2.15. *For any integer n, $p(48037937n + 1122838) \equiv 0 \bmod 17$.*

Notice that $48037937 = 17 \cdot 41^4$. Isn't math crazy? It turns out all of this can be generalized.

Theorem 2.16 (Ono). *For any integer p greater than 5 that is relatively prime with 6, there exists some integers a, b such that $p(an + b) \equiv 0 \bmod p$ for any integer n.*

However, this is still open for general integers. In particular, the following conjectures are still open.

Conjecture 2.17 (Newman). If k is a positive integer, then in every residue class $r \bmod k$, there are infinitely many integers n such that $p(n) \equiv r \bmod k$.

Conjecture 2.18 (Erdos). If p is prime, there exists at least one integer $n > 0$ such that $p(n) \equiv 0 \bmod p$.

In sum, the study of partitions is a rich field, and we have illuminated just a few aspects of it here. If you are interested in further topics in this field, try your hand at some of the practice problems below, as well as reading about Young tableaux, diagonal partitions, plane partitions, and some further results in the field that deal with different representations of then, restrictions on them, and higher dimensional version of them.

2.6 Practice and Challenge Problems

Note that, as a hint, most of these problems can be solved using the generating functions techniques we described earlier.

2.6.1 Basic Problems

Problem 2.6.1 (Notes of Yufei Zhao)**.** *Let n be a positive integer. Show that the number of partitions of n into parts which have at most one of each distinct even part equals the number of partitions of n into which each part can appear at most three times.*

Problem 2.6.2. *Show that the number of partitions of n with no part equal to 1 is $p(n) - p(n-1)$.*

Problem 2.6.3. *Let n be a positive integer. Show that the number of partitions of n, where each part appears at least twice, is equal to the number of partitions of n into parts all of which are divisible by 2 or 3?*

Problem 2.6.4 (Peter Csikvari)**.** *Let $p(n)$ be the number of partition of n, where $p(0) = 1$. Let $s_n(i)$ be the number of i's in all partitions of n.*

(a) Show that $s_n(1) = \sum_{k=0}^{n-1} p(k)$.

(b) Find similar formulae for $s_n(k)$ for a general k.

(c) Use the ideas from (a) and (b) to show that

$$p(n) = \frac{1}{n} \sum_{k=1}^{n} \sigma(k) p(n-k),$$

where $\sigma(k)$ is the sum of the positive divisors of k.

Problem 2.6.5 (Peter Csikvari)**.** *Let $c(p)$ be the number of distinct parts of a partition p, and let P denote all of the partitions of n. Show that $\sum_{p \in P} c(p) = \sum_{i=0}^{n-1} p(i)$.*

Problem 2.6.6. *Let n be a positive integer. Show that the number of partitions of n into odd parts greater than 1 is equal to the number of partitions of n into unequal parts none of which is a power of two.*

2.6.2 Competition Problems

Problem 2.6.7 (Putnam 1957)**.** *Let $\alpha(n)$ be the number of representations of a positive integer n as a sum of 1's and 2's, where order matters. Let $\beta(n)$ be the number of representations of n as a sum of integers greater than 1, again where order matters. Show that $\alpha(n) = \beta(n+2)$.*

Problem 2.6.8 (Putnam 2003)**.** *Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + \cdots + a_k$, such that $a_1 \le a_2 \le \cdots \le a_1 + 1$? For example, with $n = 4$, there are four ways: 4, $2+2$, $1+1+2$, $1+1+1+1$.*

2.6.3 Research Problems

There are myriad open conjectures, some of which we've discussed, in partition theory. One possible research problem in the area[2] involves a little bit of description. First, we'll be wanting to examine the number of partitions of n into powers of 2, none of which are repeated more than twice. Let b_n denote this number. It turns out that these have a connection to complete binary trees and the countability of the rationals, a link that you have research on your own. Either way, from inspection, we can find that it seems that every third term in this sequence is even and the rest are odd; one open problem is to prove this.

Problem 2.6.9. *Show that* $b_n \equiv 0 \bmod 2$ *if* $n \equiv 0 \bmod 3$*, and is* $1 \bmod 2$ *otherwise. Does a similar result hold for other primes other than* 3*?*

There are several other open problems related to sequences constructed from this sequence, as well, as related sequences derived from powers of three repeated up to three times, and in general powers of a prime p repeated up to p times. Feel free to look into these on your own and make some conjectures, if not find a new result! You never know what you might find in the wondrous world of partitions!

[2] As described in a Clemson math research program document, which can be found here `http://www.math.clemson.edu/~kevja/REU/PossibleProblems.pdf`