Generating Functions and Linear Recurrences

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In addition to the other useful applications, you can also use generating functions to solve linear recursive equations! We did this when we worked on finding the generating function for the Fibonacci Numbers.

For example: Let $a_0 = 1$ and let $a_n = 2a_{n-1} + 1$ Find a simple formula for a_n.

If we write out a list of the first couple of terms, we find

$$1, 3, 7, 15, 31, 63, ...$$

We could try and find a pattern from here, but lets use generating functions. Let $f(x)$ be our generating function for this sequence. Thus,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... = 1 + 3x + 7x^2 + 15x^3 + 31x^4 + 63x^5 + \dots$$

Notice that

$$xf(x) = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + ... = 1x + 3x^2 + 7x^3 + 15x^4 + 31x^5 + 63x^6 + \dots$$

Because $a_n = 2a_{n-1} + 1$, we see that $a_n - 2a_{n-1} = 1$, so

$$\begin{aligned}
f(x) - 2xf(x) &= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1)x^2 + (a_3 - 2a_2)x^3 + (a_4 - 2a_3)x^4 + \dots \\
&= a_0 + (1)x + (1)x^2 + (1)x^3 + (1)x^4 + (1)x^5 + \dots \\
&= 1 + x + x^2 + x^3 + x^4 + x^5 + \dots \\
&= \frac{1}{1-x} \\
\Rightarrow f(x)(1 - 2x) &= \frac{1}{1-x} \\
\Rightarrow f(x) &= \frac{1}{(1-x)(1-2x)}
\end{aligned}$$

Now that we have solved for $f(x)$, we want to try and see if we can somehow recover what the coefficients will be when we expand it out. If we can, then we can find a general formula for a_n. We want to use the geometric series formula we used before, but the denominator of this fraction is not very conducive to doing so. Lets try to break it up using partial fractions.

$$\begin{aligned}
\frac{1}{(1-x)(1-2x)} &= \frac{A}{1-x} + \frac{B}{1-2x} \\
1 &= A(1-2x) + B(1-x) \\
&\Rightarrow A = -1 \\
&\Rightarrow B = 2 \\
\Rightarrow \frac{1}{(1-x)(1-2x)} &= \frac{2}{1-2x} - \frac{1}{1-x}
\end{aligned}$$

We know what the expansion of the second fraction is: $1 + x + x^2 + x^3 + x^4 + \dots$. Now we need to figure out what the expansion of the first one is.

$$\begin{aligned}
\frac{2}{1-2x} &= 2 \cdot \frac{1}{1-2x} \\
&= 2 \cdot \frac{1}{1-(2x)} \\
&= 2 \cdot (1 + 1(2x) + 1(2x)^2 + 1(2x)^3 + 1(2x)^4 + \dots) \\
&= 2 \cdot (2^0 + 2^1x^1 + 2^2x^2 + 2^3x^3 + 2^4x^4 + \dots + 2^nx^n + \dots) \\
&= 2^1 + 2^2x^1 + 2^3x^2 + 2^4x^3 + 2^5x^4 + \dots + 2^{n+1}x^n + \dots
\end{aligned}$$

Thus,

$$\begin{aligned}
f(x) &= \frac{2}{1-2x} - \frac{1}{1-x} \\
&= (2^1 + 2^2x^1 + 2^3x^2 + 2^4x^3 + 2^5x^4 + \dots) - (1 + x + x^2 + x^3 + x^4 + \dots) \\
&= (2^1-1)x^0 + (2^2-1)x^1 + (2^3-1)x^2 + (2^4-1)x^3 + (2^5-1)x^4 + \dots + (2^{n+1}-1)x^n + \dots
\end{aligned}$$

From here, we can see that we have recovered the original coefficients, but in a general form. Thus, $a_n = 2^{n+1} - 1$.

For this problem, there are many other methods (including just seeing the pattern from the list of numbers) that may have been easier, but this is still an interesting approach. The good thing about the generatingfunctionological (yes, it is a word :P) approach to this sort of problem, is that even when the numbers get messier or the problem more complicated, we can always resort to bashing algebra.

Here are some practice problems for you to try. Find a simple formula for the nth term of the sequence using generating functions. a) $a_0 = 1$ and $a_n = 3a_{n-1} + 2$ b) $b_0 = 2$ and $b_n = 4b_{n-1} - 3$ c) $c_0 = -1$ and $c_n = -2c_{n-1} + 1$

If you've finished the above three problems and want something a little more difficult, try proving it in general!

Find a formula for the nth term of the following recurrence.

$$a_0 = k$$

$$a_n = pa_{n-1} + q$$

This generatingfunctionological approach also works for recurrences of the form $a_n = pa_{n-1} + qa_{n-2} + r$ (like the fibonacci numbers), it is just a little more messy. This process can be extended to have any number of a_{n-k}s as you want!