1. Notes from lecture 1

Remark 1. We use the following notation:

(1) $\mathbb{Q}$ set of rational numbers
(2) $\mathbb{R}$ set of real numbers
(3) Let E be a subset of an ordered set X, that is $E \subset X$. Then if $x \in X$ is the least upper bound of E, we say:

$$x = \sup E$$

that you should read as "x is the supremum of E". From the other hand, if $x \in X$ is the greatest lower bound of E, we say:

$$x = \inf E$$

that you should read as "x is the infimum of E".

Definition 2. Given two sets X and Y, the Cartesian product of X and Y is the set of all ordered pairs of elements of X and Y, that is:

$$X \times Y = \{(x, y) \text{ s.t. } x \in X, y \in Y\}$$

Definition. Given two sets X and Y, a binary relation $\mathcal{R}$ over X and Y is a subset of the Cartesian product $X \times Y$, that is:

$$\mathcal{R} \subset X \times Y$$

If we specify just one set, say X, we use the convention $\mathcal{R} \subset X \times X$

- If $x \in X$ and $y \in Y$, we say that x is related to y, in symbols $x\mathcal{R}y$, if and only if:

$$(x, y) \in \mathcal{R}$$

- We say that x and y are not related, in symbols $x\neg\mathcal{R}y$, if and only if:

$$(x, y) \notin \mathcal{R}$$

I shall give you an alternative definition of order which you might find easier to work with.

Definition 3. Given a set X, an **order** relation $\mathcal{R}$ on X is a binary relation such that

(1) For any $a \in X$, $a\mathcal{R}a$
(2) For any $a, b \in X$, if $a\mathcal{R}b$ and $b\mathcal{R}a$ then $a = b$
(3) For any $a, b, c \in X$, if $a\mathcal{R}b$ and $b\mathcal{R}c$ then $a\mathcal{R}c$

Example 4. Let $X = \mathbb{R}$ and consider a binary relation $\mathcal{R}$ over X defined as follows:

$$\text{For any } a, b \in X,\ a\mathcal{R}b \text{ if and only if } a - b \text{ is a non negative number}$$

In this particular case, $\mathcal{R}$ coincides with the usual notion that you have of $\geq$, that is "greater or equal to". Let's verify that $\geq$ is indeed an order relation. It has to satisfy the three proprieties that I have just mentioned

(1) Clearly for any real number $x \in \mathbb{R}$, we have $x \geq x$. So $x\mathcal{R}x$ and the property is satisfied
(2) If $x \geq y$ and $y \geq x$ then the only possibility is that $x = y$. Also the second property is satisfied

(3) If $x \geq y$ and $y \geq c$ then it must be $x \geq c$. Also the third property is satisfied. Hence $\geq$ is an order relation

Remark 5. In class, I have indicated a general order relation with the symbol $>$. This should not be misleading. In this context, you should not think of it as "strictly greater of". It is just a relation $\mathcal{R}$.

Definition. Let E be a subset of an ordered set X, that is $E \subset X$.

- Then $x \in X$ is the **least upper bound** of E if and only if:
  (1) x is an upper bound for E
  (2) If $y < x$ and $y \in X$, then y is not an upper bound for E.
- From the other hand, $x \in X$ is the **greatest lower bound** of E if and only if:
  (1) x is a lower bound bound for E
  (2) If $y > x$ and $y \in X$, then y is not a lower bound for E.

2. Problems

Problem 6. Let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$.

(1) Write down the elements of $A \times B$.
(2) You can think of $A \times B$ as a set whose elements are ordered pairs. Let $C = A \times B$. Can you give an example of an order relation $\mathcal{R}$ over C?

Problem 7. Let $X = \mathbb{R}$ and E be a subset of X, that is $E \subset X$. We define E as follows:

$$E = \left\{ q \in \mathbb{Q} \, \text{s.t.} \, q < \sqrt{2} \right\}$$

where $<$ means "strictly smaller than".

(1) Give an example of upper bound for E. Is it unique?
(2) What's the least upper bound of E? Show that it satisfies the definition of supremum.
(3) What's the maximal element of E?
(4) Give an example of lower bound for E. Is it unique?
(5) What's the greatest lower bound of E? Show that it satisfies the definition of infimum.
(6) What's the smallest element of E?

Now, let X be the set of rational numbers, that is $X = \mathbb{Q}$. Answer again questions (1) to (6)

Problem 8. Is the least upper bound unique (given that it exists) ? Prove it.