INTRODUCTION TO REAL ANALYSIS

1. Syllabus

- Ordered sets
- Metric spaces
- Compact sets
- Limit of sequences
- Series
- Continuity

2. Application Questions

(1) Rewrite the following statements using just mathematical symbols (e.g., $\forall, \exists, \in, \notin, \subset, \supset, \Rightarrow, \mathbb{Q}, \mathbb{R}$) :
   (a) "For any element x of the set X there exists an element y of the set Y such that the sum of x and y belongs to the set Z"
   (b) "The sum of any two rational numbers is also a rational"
   (c) "Any real number admits an inverse element"

(2) Given the sets of integers:

$$A = \{2, 4, 5, 6, 9\} \qquad B = \{x \in \mathbb{N} : x + 2^x < 70\}$$

list the elements of the following sets:
   (a) $A \cup B$
   (b) $A \cap B^c$
   (c) The set of all possible subsets of $A \cap B^c$
   (d) $B \setminus A$
   (e) The Cartesian product of $B \setminus A$ and $A \setminus B$

(3) Explain briefly your answer to the following questions:
   (a) Find $p \in \mathbb{Q}$ such that:

$$p^2 = 2$$

   (b) What is the smallest element of this set?

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

(4) We adopt the following notation:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Let:

$$A_i = \left\{ x \in \mathbb{R} : 1 - \frac{1}{i+1} < x < 1 + \frac{1}{i+1} \right\}$$

What is $\bigcap_{i=1}^{n} A_i$ then?

Now assume that n is greater than any integer. Can you guess what $\bigcap_{i=1}^{n} A_i$ is in this case?

(5) Assume that a function $f : \mathbb{R} \to \mathbb{R}$ has the following property:

$$\forall \epsilon > 0\, \exists \delta > 0 \,\text{s.t.}\, x \in \mathbb{R},\, |x-2| < \delta \Rightarrow |f(x) - 1| < \epsilon$$

what can you say about the value of $f(2)$?

How does your answer change if we slightly modify the property as follow?

$$\forall \epsilon > 0\, \exists \delta > 0 \,\text{s.t.}\, x \in \mathbb{R},\, 0 < |x-2| < \delta \Rightarrow |f(x) - 1| < \epsilon$$