General Relativity in a Nutshell

June 26, 2011

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1 Preface

These Notes are heuristic invitations to the concepts about spacetime that lead to the formulation of general relativity, as well as some of its consequences. They are mathematically unrigorous, and are not afraid to "lie" to keep things simple. I have kept prerequisites to the absolute minimum: familiarity with cartesian coordinate euclidean geometry, and some basic knowledge of high school physics would help. For a more mathematically correct and systematic build up for the subject, a much longer work is needed. Some good books to learn general relativity in increasing order of sophistication are:

- Exploring Black Holes
- Schutz, Introduction to General Relativity
- Introduction to General Relativity, Ryder (the most explicit book I've found so far on the calculational part)
- Carroll, Spacetime and Geometry (introduces a lot of fancy mathematical tools, I didn't find them too useful when I learnt from Carroll, but I might in the future)
- Gravitation, by Misner, Thorne, Wheeler (HUGE reference book, with a lot of nice pictures and very exhaustive explanations)
- Wald, General Relativity (Mathematically sophisticated, concise and very correct, but at the cost of being very intimidating. Carroll is basically a mathematically 'softer' translation of Wald. The contents can almost be isomorphically mapped between the two)

These notes are not to be sold, and I do not claim ownership on any of the figures I recklessly pillaged on the internet, or the ideas presented.

2 Introduction

General Relativity can be summarized in the two following statements:

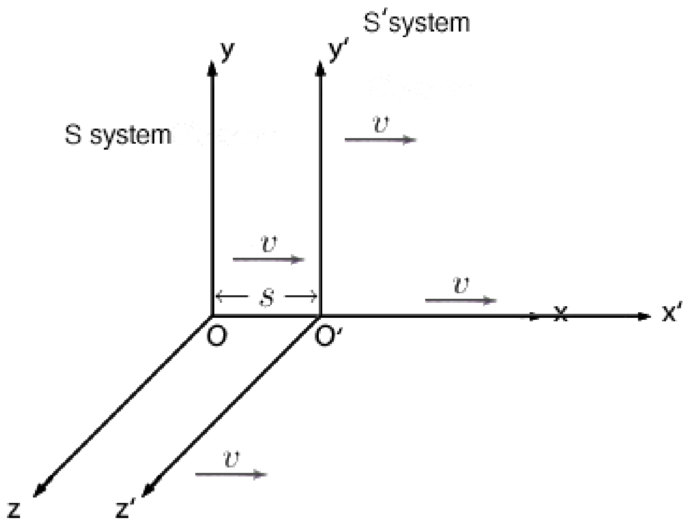
- Matter tells spacetime how to curve
- Spacetime tells matter how to move

In this talk, I will try to convince you that the two previous statements form a logical and consistent way to describe gravity, leading to very nontrivial consequences like black holes, big bang, etc...

2.1 Pre-Relativity concepts

Before Einstein, people thought of space at a fixed point in time. This means that an event happening at a time t could be described by its coordinates:

$$(x, y, z, t) \tag{1}$$



$$
\left.\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}\right\}\ \text{Galilean Transformation Equations}
$$

$$
\left.\begin{aligned}
x &= x' + vt' \\
y &= y' \\
z &= z' \\
t &= t'
\end{aligned}\right\}\ \text{Galilean Inverse Transformation Equations}
$$

If we switch to a frame moving with velocity

$$v = (v_x, v_y, v_z) \tag{2}$$

Then intuitively, the new coordinates viewed from the moving observer would be:

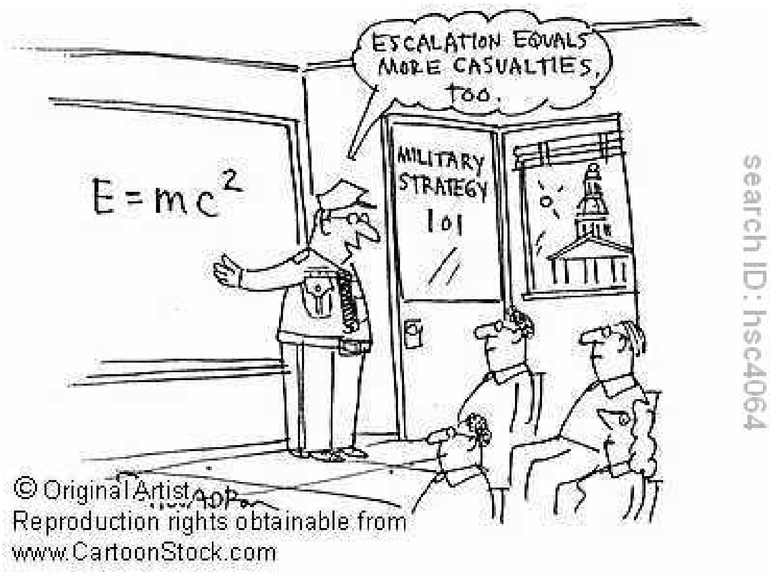
$$(x', y', z', t') = (x - v_x t, y - v_y t, z - v_z t, t) \tag{3}$$

Those are called **Galilean Transformations**. Note that only the spatial coordinates change, the time coordinate is fixed ($t' = t$), which means time flows steadily the same what between the different frames.

CheckPoint

- If I shoot a light beam a speed c, and you move at speed v relative to me, how fast do you see the light beam propagating?

2.2 There comes Einstein...



But light propagates at a fixed speed c in ANY FRAME...

Galilean transformations are inconsistent with that fact. To make light propagate at a fixed speed in any frame, we need to describe events from a certain inertial observe by the 4 spacetime coordinates (x, y, z, t). Let us write (x,t) as the principle coordinates to make the math simpler.

Suppose a certain event, call it event A, happens at

$$A = (x, y, z, t)$$

. For example you can think of A as the event of me clapping my hands at spatial coordinates (2, 3, 4) at time 5PM:

$$A = (2, 3, 4, 5PM)$$

Suppose another observer, who moves at speed v in the $+x$ direction, wants to describe the same event A, The transformation that is consistent with the constant speed of light for both observers is called a **Lorentz Transformation**, pictured below:

$$(x, t) \to \gamma(x - vt, t - \frac{vx}{c^2}) \tag{4}$$

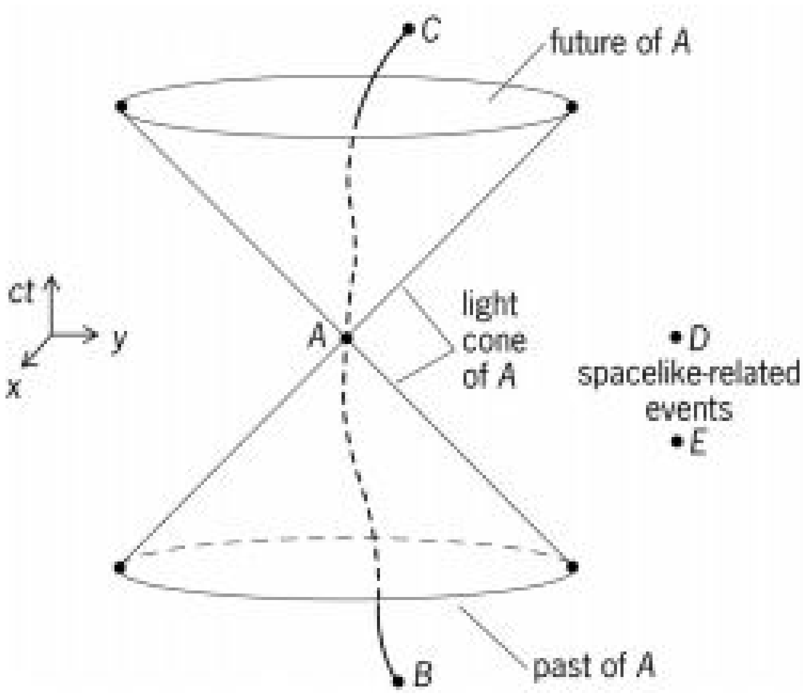
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5}$$

We define the **worldline** of a particle as the trajectory it traces through spacetime. Note that the worldline has to be inside the forward lightcone, because NOTHING, moves faster than light. The **forward light cone** is a boundary which in spacetime separates all the points that you can reach in your future (points that are not farther apart than the distance light can reach), from the points you can't.

CheckPoint : Check that if you shoot a light beam at speed c, another observer still sees the light beam propagating a speed c. Check that you cannot outrun a light beam.

Results of SR recap

- Inertial Frame extend throughout spacetime
- Absolute speed of light in every inertial frame implies all interactions must be slower or equal to light propagation, leads to light-cone description and Lorentz Transformation
- $E = mc^2$



2.3 Gravity is different

Newtonian Gravity : In an inertial frame, gravity is experienced as a force that makes objects accelerate ($F = -\frac{GMm}{r^2} = ma$), and is experienced **instantaneously** from source to observer.

Since special relativity forbids instantaneous propagation of interaction, we need to modify this.

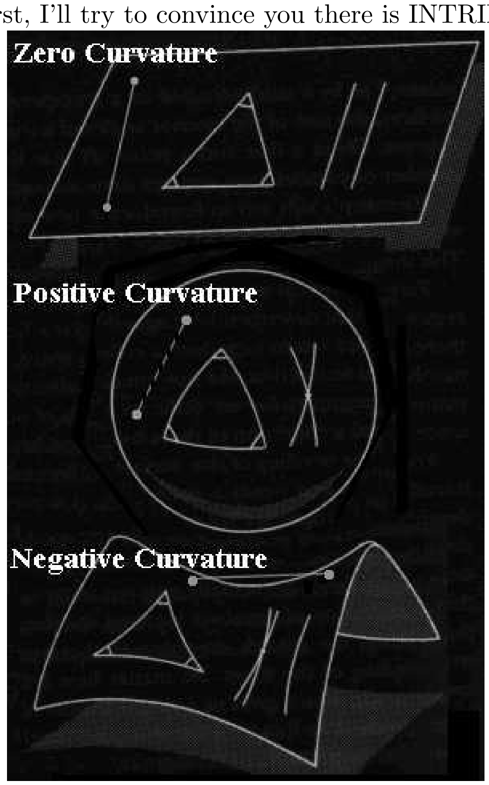
Possible reformulation? If you had a course in physics, you'll quickly notice that the gravitational force looks exactly like the electromagnetic force. So if electromagnetism is consistent with gravity, why not write a theory of gravity that looks exactly like the maxwell's equations, which describe electromagnetism AND are consistent with special relativity?

Why Gravity is different Gravity affects everything in the same way (same acceleration for all objects). Some people might refer to this as inertial mass = gravitational mass. This is different from electromagnetism which accelerates things based on their charges. This means it is more elegant to formulate gravity as a geometric theory rather than a force. This is because geometry affects everything the same way. The consequence of having the same effect on all masses is that gravity cannot be detected locally. This is the **principle of equivalence**.

3 An invitation to curvature

3.1 Some Heuristic Examples

First, I'll try to convince you there is INTRINSICALLY different about a curved space



CheckPoint :

For the following, give the appropriate description of curvature:

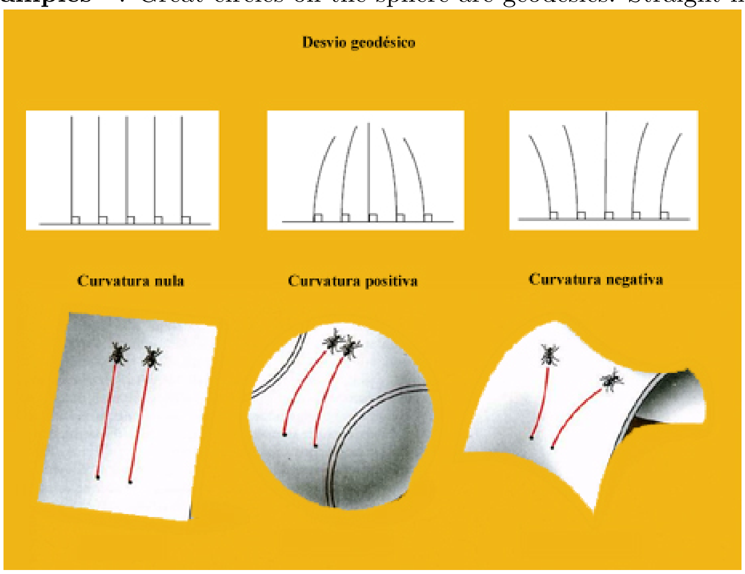
- Is the surface of a cylinder curved?
- Is the surface of the Sphere curved?
- Is the surface of a Hyperboloid sheet curved?

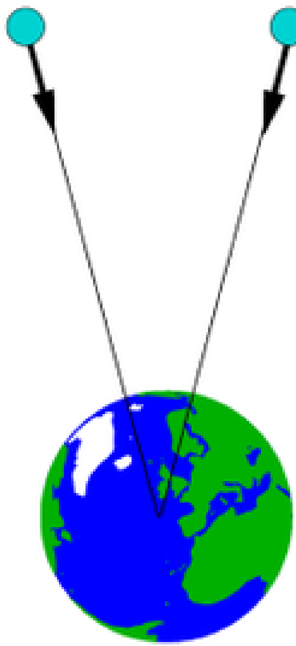
Misconception : Note that local description of curvature gives NO information on the topology of the object.

3.2 Observing Curvature: Geodesic Deviation

Definition : A Geodesic is the line element on a space which has the shortest path between its endpoints.

Examples : Great circles on the sphere are geodesics. Straight lights on a plane are geodesics.

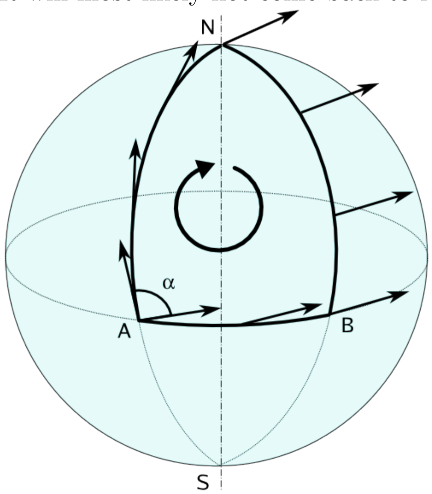




3.3 Observing Curvature: Parallel Transport

What makes a space curved at a certain point? Well, we know the surface of a sphere is curved, thanks to the analogy developed by ryder. Now take an arrow, stick it at the north pole, and parallel transport it around a loop. By **parallel transporting**, what I mean is keep it locally tangent to the direction you move on the loop.

It will most likely not come back to itself. The change in the angle measures how "curved" the space is.



Now, you want to describe the curvature locally. Well, just make the loop smaller and smaller....

Quick Aside: The Riemann Curvature Tensor :

A tensor is a collection of numbers. Here is a sketch why curvature needs, several numbers, i.e. a Tensor, and not one, to encode it. Suppose you want to tell everything about the curvature of a 2 dimensional surface locally. How many parameters do you need? It is the number of independent ways you can draw loops and choose arrows to parallel transport (NOTE: this is wrong technically and in actuality, because some of those numbers might be redundant)

There are 2 parameters to specify a 2-D arrow. There are 4 parameters for a loop (2 vectors, each with 2 components define a loop uniquely). So a total of 8 parameters to specify the starting point of the experiment. The change in the vector is encoded by 2 parameters (the vector of change). Hence, you have a total of $2 * 4 * 2 = 16 = 2^4$ parameters. The object that has all those 16 parameters is called the **Riemann curvature Tensor**. You will see it later in Einstein's equation, encoded in the symbol R.

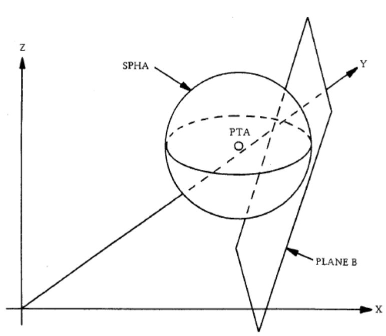
: The number of parameters to specify curvature increase considerably in 4 dimensional spacetime. For a d dimensional spacetime, I claim you have d^4 number of parameters that specify how to transport a vector in a small loop. The Riemann curvature tensor in 4-d spacetime has $4^4 = 256$ components at 1 location! Fortunately, only 20 of those components are independent.

3.4 What makes curved space curved? (mathy version)

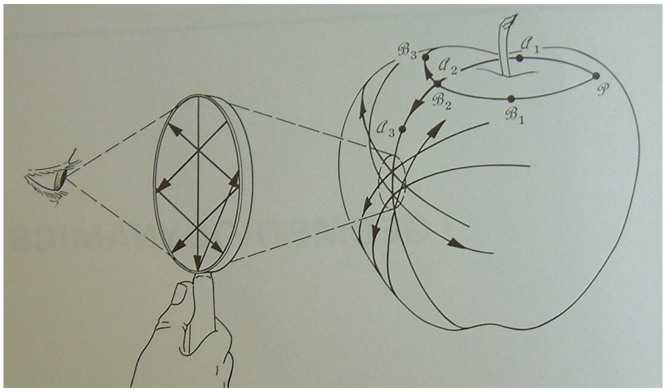
The geometry of a space is described by its curvature. But what is curvature? How do you know some surface is curved? For example, how do you know the surface of a sphere is curved?

What a stupid question, isn't it? You'll just tell me you look at the 2 dimensional surface of a sphere in a 3 dimensional space, compare it with a flat space you know (a plane), then say, they are not the same! It turns out that the technique you just described is called **embedding**. Formally speaking, what you did was you **embedded** the 2 dimensional surface, for example the surface of a sphere, into a higher dimensional space that is **flat**, euclidean space in this case. In this case, you know the 3 D space is flat because distances are described by

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{6}$$



3.5 The story of the ant (mathy version)



Aha.. this was too easy for you right? Let me make the question a little bit harder. You are now an ant, walking on an sphere surface. How do you know the surface of the sphere is curved? There's actually several ways you can go about doing this, I'll describe one to you.

The metric It turns out there is a solution for that too. Here's what you do. You draw an grid on the apple, where the grid assigns every point on the surface of the apple to 2 numbers, the coordinates (θ, ϕ) depicted in the picture below.

Then you start collecting a table measuring the distance with your tiny ruler between all nearby points of point A, point B , point C... until you exhaust all the points on the surface of the apple.

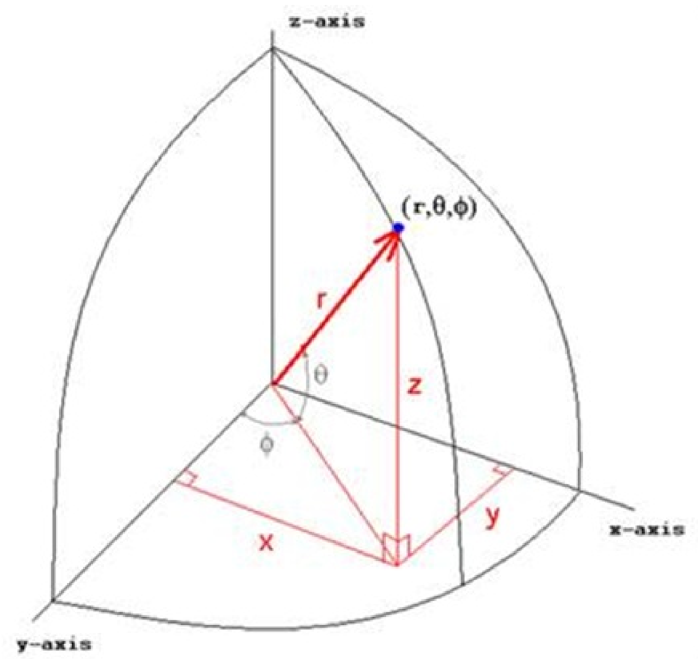
You end up with a HUGE table of values.You can write a formula that works locally to describe all the distances near that point.

If the ant learnt about spherical coordinates, it would quickly realize that local distances could be expressed as:

$$\Delta s^2 \approx R^2(\Delta\phi^2 + \sin^2\theta\Delta\theta^2) \tag{7}$$

Bravo! You just described the geometry **intrinsically**, that is without embedding it into a higher dimensional space; you have encoded the information in what is a matrix called the **metric** denoted g:

$$g = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$



It turns out that there is something intrinsically special about the metric g that makes it describe a curved space. It turns out that **regardless of what coordinates you use** to label the points, there is NO way you can turn g everywhere on the surface of the sphere into:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This, my friend, is why curved the surface of the sphere is different from the flat surface described in polar coordinates, why curved space is curved.

(end of mathematical rant)

3.6 Physical Consequence of Curvature

We claimed that we could formulate gravity as a consequence of the curvature of spacetime. Here's how. We have seen that curvature implies geodesic deviation. This is a physical effect that can be observed.

Tidal Force : Suppose you are falling towards the earth with 2 marbles. How do you know you are in a gravitational field? Suppose you are a tall person. How do you know you are a gravitational field?

Gravitational Redshift :

Send two signals of light at 2 different times through a gravitational field. They the two signals will be received at a larger/shorter time interval than it was emitted.

4 Gravity as Geometry

4.1 In Words

What we did above was math. Here we'll get to the physics.

There is 2 equations that describe general relativity. The first one is the geodesic equation. Here it is in the version we learnt:

$$\Delta s^2 = minimumpossible \tag{8}$$

It just says particle move in spacetime on path that minimizes the real distance measured. Wow, like we didn't know that already. But there's an additional input that makes the trajectories interesting. If the spacetime in which the particle was moving was flat, the lines would not geodesically deviate, right? You would not be able to observe gravity...

Now the 2nd equation makes everything different:

$$Curvature = energy/momentum \tag{9}$$

This equation says that matter that moves through space time, will have energy and momentum. That energy-momentum in turn dictates what the curvature of spacetime looks like at that point. But curvature, in turns make trajectories geodesically deviate...

So basically the two equations above say the following:

- Matter-motion tells spacetime to curve
- (Curved) Spacetime tells matter how to move

Combined together is all the beauty of gravity

4.2 Quick Aside

Why is general relativity is called "nonlinear"?

I'll separate the nonlinearity in 2 parts.

Gravity is different from Electromagnetism Electromagnetism as formulated by Maxwell is a linear theory. It means that the solutions to the force by 2 charges is related by adding the solutions by each charge separately. This is not true for the theory of gravity.

Gravity Couples to itself The right hand side of einstein's field equations is energy and momentum. This means that energy is a source of curvature, or gravitational effect. But how is that gravitational effect transmitted? We can work by analogy. In the quantum theory of electromagnetism, electromagnetic effects are transmitted by photons, little packets of energy that travel through spacetime. Electromagnetism is linear, in a way that those photons don't interact with each other to a very good approximation. They only interact with charged object, and that interaction is what you observe as electromagnetic forces. If you linearize (approximate) Einstein's field equations, you'll get equations that don't look too different from Maxwell's equations that describe electromagnetism. In that limit, you also have solutions for propagating energy packets that can transmit gravitational effect (which we call gravitons), if you write its quantum version. However, the right hand side of the field equations ensures that the graviton, which has energy will interact with other gravitons. Why? Because a graviton has energy, and hence will curve spacetime. This spacetime, in turn modifies the trajectory of the other graviton.

4.3 General Relativity in Equations

Here are exactly the two above equations, written in correct/standard mathematics:

$$\frac{d^2x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0 \tag{10}$$

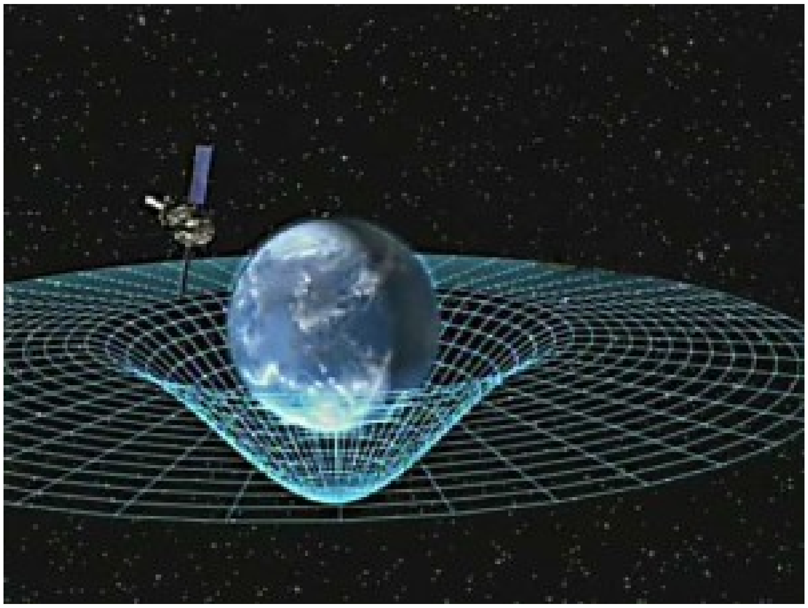
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,R = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{11}$$

Equation 17 corresponds to equation 15, and is called the geodesic equation. Equation 18 corresponds to equation 16, and is called "Einstein Field Equations" Note, that it is plural, in fact, it is a set of 16 equations. Just needed to put it in here, for you guys to be blessed with this magnificent view... OKAY, moving on.

If you watch carefully, you actually see an object called R in equation 18. It is actually a sized down version of the 256 component Riemann Curvature Tensor, described in the mathy version of the curvature section. The object g is the metric, which we encountered before, and the stuff called $\frac{8\pi G}{c^4}$ is just a constant. The thing called $T_{\mu\nu}$ is actually the energy-momentum tensor, a 16 number object that encodes everything you need to know about how much matter and what matter is doing at a certain point in spacetime.

Checkpoint

- Why does light bend in a gravitational field?
- What is gravitational reshift?



5 Black Holes

5.1 Non-rotating Black Holes

Let's review back to what we said about forward light cones. Forward light cones describe all the possible points of your future. Now here's a scenario for you to visualize. Suppose you have a object that is SUPER DUPER dense. It is so dense, in fact, that there is a radius **outside** of the object in which your forward light cone is so bent, that it actually points radially inward instead of forward in time. Note, gravity bends spacetime, but your forward light cone is a surface in spacetime, so gravity will bend your forward light cone!

Derivation How dense does the mass have to be? We can make a heuristic derivation of the amount of stuff you need to pack within its **Schwarzschild Radius** (R_s) in order to make a black hole. We apply conservation of energy and set the initial velocity as the light velocity

$$\frac{1}{2}mv^2 - \frac{GMm}{R_s} = E = 0 \tag{12}$$

$$v = c \tag{13}$$

$$R_s = \sqrt{\frac{2GM}{c^2}} \tag{14}$$

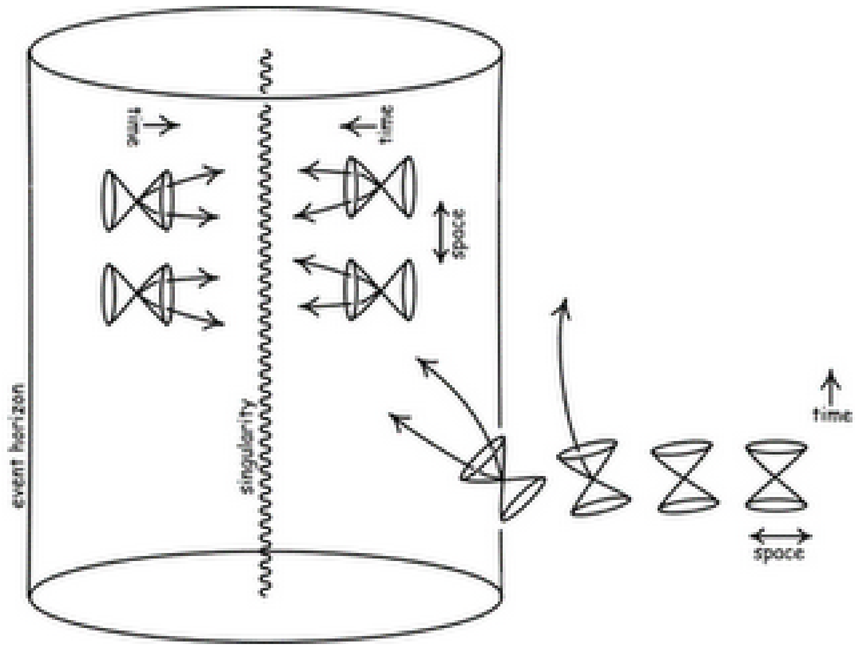
What's the consequence :

Well think about this. In flat space, your forward light cone FORCES you to move forward in time. Now, if the forward light cone points inward radially instead of begin in the time direction, oh oh... you are big trouble. That means your worldline (your trajectory through spacetime) will necessarily point inward radially. Even light, which moves ON the forward light cone must point radially.

- **consequence 1**: This means, well, even LIGHT cannot escape from that region of gravity. That's why such an object is called a black hole. If light within that region does not escape the region, an observer far away can't SEE the object. Effectively, it looks black.

- **consequence 2**: If moving forward in time now becomes moving inward radially, you will not be able to escape regardless of how hard you struggle... Since you cannot outrun light, and light cannot escape, that tells me you are in big trouble.

  Terminology: The surface within which your forward light cone points inward radially is called the **Event Horizon**. The name comes from the fact that light is infinitely stretched out close to that surface which means that if you watch someone fall into a black hole, you NEVER see them cross the surface. It is effectively the horizon of what you can observe outside.



Correcting Misconceptions

- The gravitational effects of the black hole are indistinguishable from the gravitational effect of a spherically symmetric object (the sun for example)
- Once you cross the horizon, you don't necessarily die. You'll die only when the tidal forces that stretch your body are too strong and rip you apart (this can happen regardless of whether you are falling into a black hole or a very dense object like a neutron star)
- A black hole, according to general relativity has NO spatial extent. In other words, the solution yields a finite amount of mass allocated to a point size object. You effectively have infinite density, and infinite curvature at the location of the black hole: This is called a singularity.

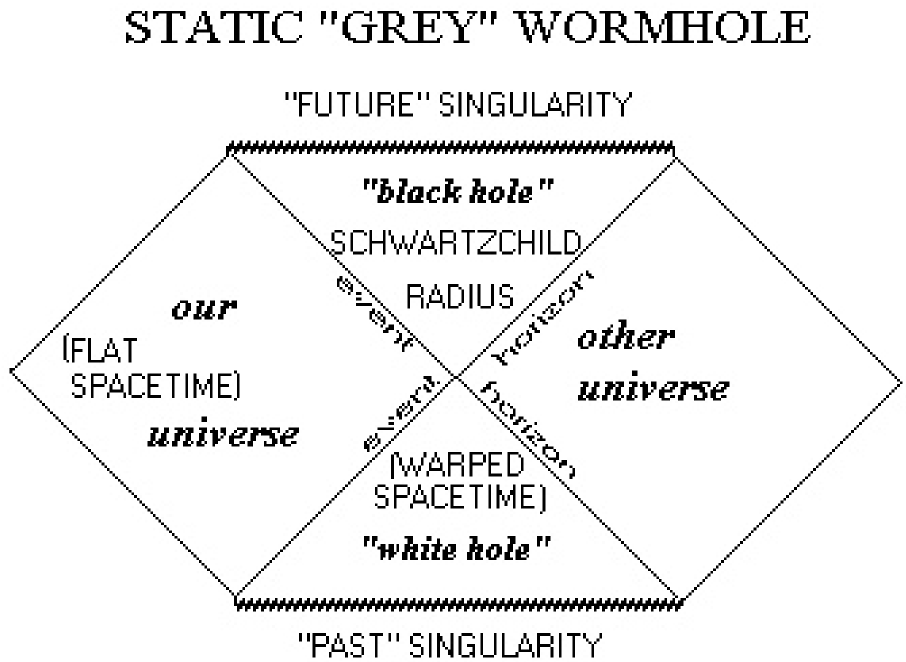
5.2 Rotating Black Holes

We saw on the right side that the thing that affects curvature is actually both energy/mass AND momentum. This means that if a black hole is rotating, it has angular momentum. That angular momentum will turn out on the right side of the equation. The consequence is that space will TWIST in the direction of rotation around a rotating black hole.

5.3 Romeo and Juliet: GR style

The FULL solution to a spherically symmetric black hole actually has 2 universes connected by the black hole itself. In other words, the 2 universes have a common region where worldlines originating from both universes can meet, aka observers from both universes can meet.

So suppose you have an imaginary girlfriend who lives in the other universe. To have a date with her, both of you guys have to cross the even horizon of the black hole that connects both universes. Great! Reunited at last. Unfortunately, once you and her crossed the horizon, there is no way for both you to escape! This story is necessarily tragic. Moral of the story is don't fall in love with imaginary girlfriends in parallel universes...



6 Cosmology

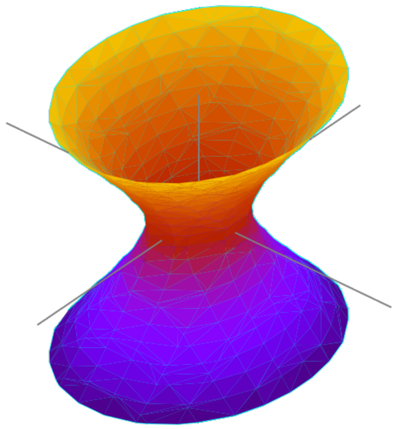
6.1 Assumptions

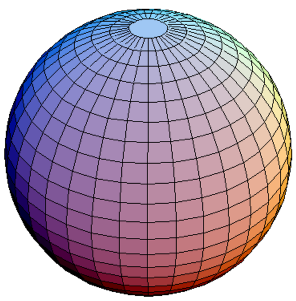
Assumptions of Cosmology:

- 1. The universe is homogeneous and isotropic in SPACE (not necessarily in time)
- 2. The physical laws at different spots in spacetime are the same.

6.2 Statics

Assumption 1 allow us to restrict the geometry of spatial hypersurface. This means that we slice the spacetime (4 dimensions) into spatial slices. Those spatial slices can look like only 1 of 3 things: a Hyperbola, a sphere, or a flat piece of paper (assuming the most trivial topology). This is because the curvature must be constant throughout the geometric structure in space. There's not 50 different type of objects with constant curvature.





Let us make the following remarks:

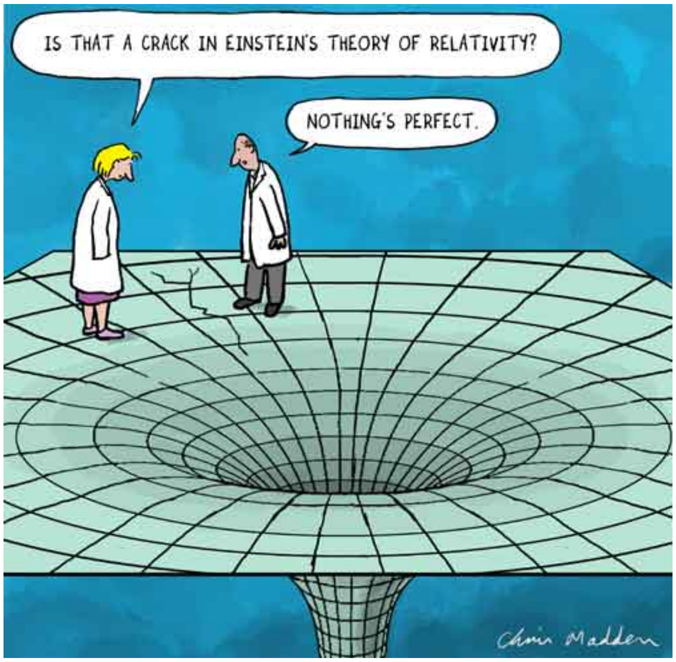
- At a given slice in time, the spatial extent of the hyperbola is infinite. That's what we mean by infinite universe (called **open universe**)
- At a given slice in time, the sphere has finite area. That means the extent of the universe at a given slice in time is finite (called **closed universe**)

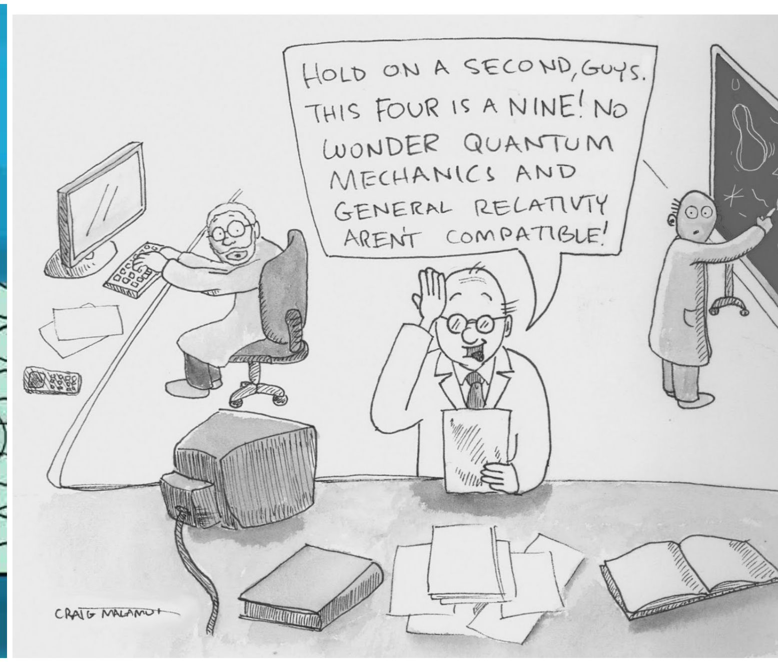
It turns out that to a very good approximation, the universe is spatially flat.

6.3 Dynamics

Now let's move to dynamics. If you solve Einstein's field equations for a isotropic universe in space, you get what are called the **Friedmann Equations** which describe the evolution in **time** of the size of the spatial slices. The evolution of the size of the spatial slices is what we mean when we say "The universe is expanding..."

7 Current Research





7.1 AdS/CFT correspondance

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AdS/CFT stands for Anti-de Sitter Space Conformal Field Theory. Anti-deSitter space is a space with constant negative curvature. Conformal Field Theory is a description (theory) of some system that has special properties (called conformal symmetries). In words, one of those properties is that your description of the system looks the same at all length scales. It turns how that theories that describe certain systems in 4 dimensions have different descriptions according to the length scale you associate with the theory. (intuitively, you can think of describing water for example. From a macroscopic point of view, it just looks like the theory of hydrodynamic waves, but if you zoom in close enough, your theory is the theory of atomic interactions). You can use the length scale of your description as an additional dimension, and visualize the theory in 5 dimensions now. When the 4 dimensional theory you are describing is scale invariant (and has additional symmetries related to conformal symmetries), the 5 dimensional description corresponds to a hyperbolic space described by 5 dimensional (quantum) gravity, in Anti-deSitter space.

7.2 Cosmological Constant

:

Why is the universe spatiallly flat? (hints of inflation) Why is there a positive cosmological constant? (dark energy) Why does most of the matter that accounts from the gravitational state of the universe not observed? (dark matter)

7.3 Big Bang and Early Universe Cosmology

What happened in the tiny nanoseconds after the big bang? In that region, both spacetime was incredibly curved, and matter incredibly hot. In this realm, laws of quantum physics and general relativity mix into something we don't really understand.

7.4 Gravitational Waves

Gravitational effects can't propagate faster than light! That means you can have jiggles in your spacetime due to oscillating sources of gravity. Those jiggles are called gravitational waves. They have not been detected yet, but Einstein's general relativity postulate they exist.