Playing with groups

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1 Definition

A group G is a set with a special element 1 and an operation $*$. We have that for any $g \in G : g1 = 1g = g$ and we also have a inverse g^{-1} such that $gg^{-1} = g^{-1}g = 1$. This immediately gives, for example, that $gh = gh' \Rightarrow h = h'$, since we can multiply on the left by g^{-1}.

2 Permutation groups

The first kind of group to consider, in some ways the most basic type of group, is a permutation group. This is, for any positive whole number n, the ways that n objects could be rearranged, we call this S_n. A somewhat subtle point is that we are not considering the ways it can be arranged, but rather the way it can ***re***arranged. The reason we are thinking about it this way is so that we can compose things, and do one after another, which makes for a much more interesting problem.

The number of ways n objects can be rearranged, as you may well know, is exactly $n!$ ways. Each S_n has a specific set of n many subgroups corresponding to S_{n-1}, obtained by all the permutations fixing a specified element (a subgroup is a subset of a group that is also a group in its own right).

3 Symmetry groups

Another type of group to consider, is for a geometrical object, the number of ways it could be put back into itself. As the name hints, the more symmetric an object is, the bigger this group will be. A simple example of this is a triangle. There are six ways to rearrange a equilateral triangle (where we allow flipping it over). Similarly, there are 8 ways to rearrange a square (it is interesting to note that this group isn't commutative). A bit harder to see is the number of ways to rearrange a tetrahedron, or a cube, not to mention something like a dodecahedron.

You'll notice quite readily that the symmetries of a triangle and square have a natural subgroup, those that don't flip the shape over. It turns out there's an analog for the three dimensional shapes, namely, there's more things that could be obtained if we allow "flipping" using the fourth dimension. We won't be doing any of that though.

It turns out the group of the triangle is in fact the same as the group S_3.

4 Subgroups

As we mentioned, a subgroup is a subset of a group that is also a group. We can define this more formally as a subset H of a group G satisfying (1) for $h, k \in H$ we have that $hk \in H$ (2) for $h \in H$ we also have $h^{-1} \in H$. You may notice that this is the definition of a group, without requiring that the identity be in the subgroup. But this is because it is already given, because for $h \in H$ we have it's inverse in H, and that their product is in H so that $hh^{-1} = 1 \in H$, and we needn't state this explicitly.

5 Order of a subgroup

One of the most basic facts to use when studying groups is that the order of a subgroup divides the order of a group. Consequentially, we have that the number of cosets of a subgroup also must divide the order of the group. A natural question to ask at this point is "what's a coset?" A coset is, given a subgroup $H \subset G$ and some element $g \in G$, the set $gH = \{gh : h \in H\}$.

First we show that cosets *partition* a group. That means that each group element is in exactly one coset. First, we certainly have, by the fact that $1 \in H$ that $a \in aH$, next we show that $a \in bH \Rightarrow b \in aH$, we have this because $a \in bH \Rightarrow a = bh$ for some $h \in H \Rightarrow b = ah^{-1} \Rightarrow b \in aH$. Lastly, we show that $aH = bH$ by showing that $a \in bH$ and $b \in cH \Rightarrow a \in cH$ (using the previous two facts, this amounts to saying that everything in aH is also in bH, and by the symmetry of the situation, this ends up saying aH and bH have exactly the same elements), we have this because we have $a = bh$ and $b = ch'$ so $a = (ch')h = c(h'h) = ch''$. So now to state it clearly, this gives as that cosets are a partition, because the intersection of two cosets is either the whole of both of them $aH = aH \cap bH = bH$, or empty $aH \cap bH = \emptyset$. This is true because if the intersection is not empty, then it contains some c, so that $c \in aH$ (by definition of being in the intersection) and then $a \in cH$ (by 2nd property), but similarly $c \in bH$ (since it's in the intersection) so $a \in bH$ (by the 3rd property) and $b \in aH$ (by second property) so that anything else in either is in both ($d \in aH \to a \in dH \Rightarrow b \in dH \Rightarrow d \in bH$). We also have that every group element is in *some* coset, namely the one it generates.

Now we show that $|aH| = |H|$ we have $|H|$ many products ah, one for each $h \in H$, so we need only show that $ah = ah' \Rightarrow h = h'$ but we have that by multiplying on the left by a^{-1}. And by definition, there's nothing else in aH. We state this result as $|G| = |H|[G : H]$, letting $[G : H]$ denote the number of cosets of H in G.

Some examples are given by the permutation group S_n. For example take S_3, this has four subgroups, one of order 3 $\{1, \sigma, \sigma^2\}$, and 3 of order 2, each given by a transposition. This satisfies the above since 2 and 3 both divide 6. As we have seen, S_n has a subgroup corresponding to S_{n-1}. But this just gives us in the above equation $n! = (n-1)!n$, which is true by the definition of factorial.

6 Cyclic subgroups

Given an element g of a finite group G, we can take its powers. Eventually, we will get back to the identity, and this gives us a subgroup, which we denote $< g >$ and call the subgroup *generated* by g. one interesting thing to note about such subgroups is that $gh = hg$ for all elements in it, which is not generally the case for groups.

Some examples of cyclic subgroups can be given by the subgroup of the symmetry group of a square or triangle fixing a face. We also will have something similar if we fix a single face of a 3d figure, or use a permutation that sends a face to another. In the permutation groups S_n, pick any permutation that shuffles around m elements where $m \leq n$, then this will be a subgroup of order m. That only stays true when we cycle things around like in a circle, however if we do disjoint cycles, we can get something cyclic bigger than n. The first example of this is in S_5, where a two cycle and 3 cycle together generate a cyclic subgroup of order 6.

7 Orbit stabilizer

A very important theorem for studying groups is the orbit stabilizer theorem. In short, it tells us how big a group is (there are some simple conditions that need to be satisfied first, but we will only be doing this in cases where it works, so we won't bother with that). If we consider a set of objects that a group operates on, we get that the size of the group is the size of the orbit of an element times the size of the stabilizer. The orbit of an element is the all the other things it can get to by applying elements of the group, and the stabilizer is the group elements that keep the element where it is (exercise: the orbits partition the set, and the stabilizer is a subgroup). We can write this as an equation as $|G| = |O_s||Z_s|$.

We can now find the size of the group of symmetries of 3d objects. The easiest way to do this is to look at the stabilizer of a vertex, side, or face. For the tetrahedron, we look at a vertex, there are 4 vertices, and the stabilizer of a vertex is the three elements which rotate around it, so that we get $4 \times 3 = 12$. For the cube, we look at a side, the stabilizer of a side has size 2, and there are 12 sides, so the group has order $2 \times 12 = 24$. For a dodecahedron, each face is a pentagon, so has stabilizer of size 5, and there are twelve faces, so the group has size $5 \times 12 = 60$.

8 Orbits and stabilizers

Playing around with orbit stabilizer is lots of fun. We can consider the set of vertices, of edges, of faces. Of pairs of opposite faces, vertices or edges. In the cube, consider opposite pairs of vertices tells us that the group of the cube is in fact S_4 in disguise. If we consider the stabilizer of a pair of opposite faces, we find that this is the group of a square.

9 Conclusion

A group studies the symmetries of an object, the more symmetrical it is, the bigger its group. The most symmetrical group we can consider is a permutation group, where we view all objects of a set of n elements as indistinguishable. Generally a group will be some subgroup of a symmetry group, which is achieved by placing some restriction on an object. This means that when an object naturally gives rise to a group, it is likely that the nature of the object won't permit certain types of permutation, and will be less than fully symmetrical. There is more than this that a group can study, but this is sort of the starting point, and actually a whole lot of what's really going on.