Introduction to Counting, Some Basic Principles

These are the class notes for week 1.

Before we begin, let me just say something about the structure of the class notes. I wrote up all of these class notes before the classes actually happened. There are a LOT of examples in here; I'm sure we won't get through all of them in class. But, what's good about that is that you can refer to new examples if you get stuck on a problem set!

The other thing that you'll notice about these notes is that there's a section reserved at the end for the *technical matters*; things that you're not necessarily responsible to know for the problem sets, but a more formal, and mathematically precise way of looking at the stuff we do. We may or may not cover that material in class. Feel free to look at it if you're interested, it may help!

Now, we begin. First, we need to understand that we are only working with whole, nonnegative numbers in this course; we have a notation for that set, and we call it the set of all natural numbers, $\mathbf{N}$. No other set can have this notation. If we deal with a function, we deal with a function from the natural numbers to the natural numbers. Let's define a function:

A relation between one set,, A, and another set, B, is said to be a **function** if for any x in A, there is one unique y in B so that $f(x) = y$. In other words, a function is like a rule that associates one number to a unique number. In this course, we're not thinking of continuous functions at all; we only care about functions from countable sets to countable sets. Sometimes we think of a function as a rule "from" one set, A, to another set B, typically notated $f : A \to B$. A function also can be called a **mapping** or a **transformation** depending on the context.

We define the factorial function by, $n! := n(n-1)(n-2)...(1)$

Example 1: Compute 4!

Solution: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.

Now, we talk about two basic principles that we will continue to use over and over throughout the rest of the course. Although surprisingly simplistic, we will see that these principles enable us to solve so many useful problems.

The Multiplication Principle: Loosely stated, the multiplication principle says that if there are a ways that something can happen, and b ways that something else can happen, then there are $a \cdot b$ ways that both of them can happen. Let's just axiomize that idea for now. The best way to see this is by example.

Example 2: Suppose that you have 3 bed sheets, 4 pillows, and 2 comforters. How many different bed arrangements are possible if you have only one bed sheet, one pillow, and one comforter at a time?

Solution: You have 3 choices for the bed sheet, 4 choices for the pillow, and 2 choices for the comforter, therefore $3 \cdot 4 \cdot 2 = 24$ choices for bed arrangements.

Example 3: How many different bed arrangements are possible if at any given time you choose two pillows?

Solution: You have 3 choices for the bed sheet, 4 choices for the *first* pillow, and *3 remaining choices for the second pillow*, and 2 choices for the comforter. Notice the difference, and notice why this makes sense. You have a total of 72 choices for bed arrangements.

Example 4: Suppose you have two cubes, each side of the cube has an asymmetric figure painted on it. Assume that no figure is the same. You toss them around, and then place them side by side. How many total possible views could you have looking at it from the front?

Solution: You have 6 choices for the side of the first cube, another 6 choices for the side of the second cube, and 4 choices for the rotation choice of the first cube, 4 choices for the rotation of the second cube, and then 2 choices for swapping the cubes left/right. If you're interested in group theory and abstract algebra, more on this at some point in your life. So, a total of $6 \cdot 6 \cdot 4 \cdot 4 \cdot 2$ ways. That's a lot of ways, isn't it?

Up until now, we've seen a lot of examples; they've all been relatively different from one another, and I hope you see what I mean about a simple idea yielding solutions to problems that you wouldn't literally want to count out all the possibilities too. See solution to Example 5, for instance.

Example 5: Create your own problem and solve it!

Example 6: You have 7 books; how many ways can you line them up on a bookshelf? (Assume right-side-up and binding facing outward for all books)

Solution: Think about the seven position as slots. You have 7 choices for the first slot, 6 for the second, 5 for the third, and so on; yielding a total of 7! ways. Let's leave answers in terms of factorials from now on; it's not necessary to know the number all the time.

Example 7: You have 5 letters labeled A, B, C, D, and E, and only 3 envelopes to put them in, labeled 1, 2, and 3. How many ways can you distribute the letters to the envelopes in distinct ways?

Solution: The only difference here is that we don't have the nice one-to-one correspondence. That's no problem though, because we can still use the multiplication principle: But, is the answer $3 \cdot 2 \cdot 1$, or is it $5 \cdot 4 \cdot 3$? Justify your answer with the multiplication principle.

And now you should be able to do the problem set involving the multiplication principle.

The Addition Principle: Now we move on to the addition principle. Again, we loosely state it: Suppose that A and B are two disjoint events (meaning that they don't have anything in common with each other) with the number of possible outcomes x and y, respectively. Then, the number of outcomes of the event "A or B" is $x + y$. Again, the best way to see this is by example. We can make this as simple or as complicated as we please.

Example 8: At a given restaurant, there are five chicken entrees and three beef entrees. How many possible entrees are there?

Solution: This is almost too embarrassing to have a solution to; eight entrees, right?

Example 9: At the same restaurant, if you get a chicken dish, you have the choice between three different sauces; if you get a beef dish, you have a choice of four different sauces. How many different entrees are possible?

Solution: Now we're getting somewhere! Now we have to use the addition principle *and* the multiplication principle. You have $5 \cdot 3 = 15$ choices for the chicken dishes, and $3 \cdot 2 = 6$ choices for the beef dishes, so therefore you have $15 + 6 = 21$ choices altogether by the addition principle. Not too bad, right?

Example 10: Suppose you have a special combination pad with the options A, B, C, 1, 2, 3, 4, 5 for your password selection. Suppose either your password contains two letters and a number, or two numbers and a letter (in that order). In other words, your password either takes the form LLN or NNL. How many different passwords are possible?

Solution: In class exercise. Please attempt this!

One thing we should note is that a counterpart of the addition principle is the **subtraction** principle. All this says is that suppose event A can happen n ways, and event B which is some subset of A can happen m ways (of course, $m \leq n$, then the number of ways that the event "A and not B" can happen is $n - m$.

Straightforward, right? You may need this principle on your homework. We will do one example.

Example 11: Refer back to Example 11. How many three-digit passwords are NOT of the form LLN or NNL?

Solution: There are a total of $8 \cdot 8 \cdot 8 = 512$ passwords in general. We just take our answer from example 11 and subtract it from 512 to get the total number of passwords NOT of the form LLN or NNL. Does this make sense?

You now have all the material you need to do the problem set. Some of the group-specific problems are more difficult than others; this will balance out over the course. If you use sources, it is OKAY, as long as you state what sources you're using when you present your solution. (These class notes don't count; you don't have to cite me. But if you're looking at a math book, it's really only fair/ethical to say that you're using them. No, I'm not going to check, but some professors in college do, so it's a good habit to get into.) Using outside sources does NOT hurt your chances at earning your group points.

Technical Matters

As promised, here is the section on technical matters.

I didn't say too much about the natural numbers, because it's almost assumed that those exist. They are beyond construction of the human mind; even some of your more intelligent animals can conceptualize some small counting numbers. However, it's important to note that from the natural numbers, we construct the integers (which include the negative counterparts,) and from there we construct the rational numbers, all fractions. After this, come the real numbers, which are extremely difficult to construct, and you'll spend about four weeks of a real analysis course constructing them if you take that course in college. Those become necessary for the study of completeness and continuous functions from the reals to the reals. But this is a counting class, so it makes sense that we are only dealing with positive whole numbers, because those are the only ones that describe whole entities. Any function we define can be considered a function from **N** to **N**.

Example 12: Suppose A=$\{1, 2, 3, 4\}$ and B=$\{5, 6, 7\}$, construct a function $f : A \to B$ such that every element is mapped to the same element. Then, construct a function such that every element is hit. Such a function is said to be **injective**.

Solution: Try it out until you get something!

We will, at some point, talk about other types of functions, including surjections and bijections. One of the problem sets down the road has a question on them (I believe it is the one for the Pigeonhole Principle.)

Note that we defined the factorial just as a product; we can actually rewrite it in a capital "pi" product notation, very similar to a summing notation where you have the capital Greek letter sigma for sum.

$$n! = \prod_{1}^{n} i$$

Furthermore, we can think of the factorial operation as a function from **N** to **N**. Just the way you would think of $f(x) = e^x$ and $g(x) = x^2 - 2x + 1$, we can think of the factorial operation as a function $!(n) = \prod_1^n i$, where the only things you're allowed to put in are natural numbers, and thus that's all you'll get out. (People have defined factorial for non-whole numbers, with the Gamma function; check it out if you know some integral calculus, it's pretty interesting).

That's about all for now, more next week when we begin with combinations and permutations of sets.