Zeta-Dimension

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Abstract

The *zeta-dimension* of a set A of positive integers is

$$\mathrm{Dim}_\zeta(A) = \inf\{s \mid \zeta_A(s) < \infty\},$$

where

$$\zeta_A(s) = \sum_{n \in A} n^{-s}.$$

Zeta-dimension serves as a fractal dimension on $\mathbb{Z}^+$ that extends naturally and usefully to discrete lattices such as $\mathbb{Z}^d$, where d is a positive integer.

This paper reviews the origins of zeta-dimension (which date to the eighteenth and nineteenth centuries) and develops its basic theory, with particular attention to its relationship with algorithmic information theory. New results presented include extended connections between zeta-dimension and classical fractal dimensions, a gale characterization of zeta-dimension, and a theorem on the zeta-dimensions of pointwise sums and products of sets of positive integers.

1 Introduction

Natural and engineered complex systems often produce structures with fractal properties. These structures may be explicitly observable (e.g., shapes of neurons or patterns created by cellular automata), or they may be implicit in the behaviors of the systems (e.g., strange attractors of dynamical systems, Brownian trajectories in financial data, or Boolean circuit complexity classes). In either case, the choice of appropriate mathematical models is crucial to understanding the systems.

Many, perhaps most, fractal structures are best modeled by classical fractal geometry [12], which provides top-down specifications of many useful fractals in Euclidean spaces and other manifolds that support continuous mathematical methods and attendant methods of numerical approximation. Classical fractal geometry also includes powerful quantitative tools, the most notable of

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which are the *fractal dimensions* (especially Hausdorff dimension [14, 12] and packing dimension [30, 29, 12]). Theoretical computer scientists have recently developed *effective* fractal dimensions [22, 20, 21, 7, 4] that work in complexity classes and other countable settings, but these, too, are best regarded as continuous, albeit effective, mathematical methods.

Some fractal structures are inherently discrete and best modeled that way. To some extent this is already true for structures created by cellular automata. For the nascent theory of nanostructure self-assembly [1, 25], the case is even more compelling. This theory models the *bottom-up* self-assembly of molecular structures. The tile assembly models that achieve this cannot be regarded as discrete approximations of continuous phenomena (as cellular automata often are), because their bottom-level units (tiles) correspond directly to discrete objects (molecules). Fractal structures assembled by such a model are best analyzed using discrete tools.

This paper concerns a discrete fractal dimension, called *zeta-dimension*, that works in discrete lattices such as $\mathbb{Z}^d$, where d is a positive integer. Curiously, although our work is motivated by twenty-first century concerns in theoretical computer science, zeta-dimension has its mathematical origins in eighteenth and nineteenth century number theory. Specifically, zeta-dimension is defined in terms of a generalization of Euler's 1737 *zeta-function* [11] $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, defined for nonnegative real s (and extended in 1859 to complex s by Riemann [24], after whom the zeta-function is now named). Moreover, this generalization can be formulated in terms of Dirichlet series [9], which were developed in 1837, and one of the most important properties of zeta-dimension (in modern terms, the entropy characterization) was proven in these terms by Cahen [5] in 1894.

Our objectives here are twofold. First, we present zeta-dimension and its basic theory, citing its origins in scattered references, but stating things in a unified framework emphasizing zeta-dimension's role as a discrete fractal dimension in theoretical computer science. Second, we present several new results on zeta-dimension and its interactions with classical fractal geometry and algorithmic information theory.

Our presentation is organized as follows. In section 2, we give an intuitive development of zeta-dimension in the positive integers. In section 3, we extend this development in a natural way to the integer lattices $\mathbb{Z}^d$, for $d \geq 1$. In addition to reviewing known properties of zeta-dimension, we prove discrete analogs of two theorems of classical fractal geometry, namely, the dimension inequalities for Cartesian products and the total disconnectivity of sets of dimension less than 1.

In section 4, we discuss relationships between zeta-dimension and classical fractal dimensions. Many discrete fractals in $\mathbb{Z}^d$ have been observed to "look like" corresponding fractals in $\mathbb{R}^d$. The most famous such correspondence is the obvious resemblance between Pascal's triangle modulo 2 and the Sierpinski triangle [28]. We show how to define "continuous versions" of a wide variety of self-similar discrete fractals, and we prove that, in such cases, the zeta-dimension of the discrete fractal is always the Hausdorff dimension of its continuous version. We also prove a result relating zeta-dimension in $\mathbb{Z}^+$ to Hausdorff dimension in the Baire space.

Section 5 concerns the relationships between zeta-dimension and algorithmic information theory. We review the Kolmogorov-Staiger characterization [34, 27] of the zeta-dimensions of computably enumerable and co-computably enumerable sets in terms of the Kolmogorov complexities (algorithmic information contents) of their elements. We prove a theorem on the zeta-dimensions of sets of positive integers that are defined in terms of the digits, or strings of digits, that can occur in the base-k expansions of their elements. Most significantly, we prove that zeta-dimension, like classical and effective fractal dimensions, can be characterized in terms of gales. Finally, we prove a theorem on the zeta-dimensions of pointwise sums and products of sets of positive integers that may have bearing on the question of which sets of natural numbers are definable by McKenzie-Wagner circuits [23].

Throughout this paper, $\log t = \log_2 t$, and $\ln t = \log_e t$.

2 Zeta-Dimension in $\mathbb{Z}^+$

A set of positive integers is generally considered to be "small" if the sum of the reciprocals of its elements is finite [2, 13]. Easily verified examples of such small sets include the set of nonnegative integer powers of 2 and the set of perfect squares. On the other hand, the divergence of the harmonic series means that the set $\mathbb{Z}^+$ of all positive integers is not small, and a celebrated theorem of Euler [11] says that the set of all prime numbers is not small either.

If a set is small in the above qualitative (yes/no) sense, we are still entitled to ask, "Exactly how small is the set?" This section concerns a natural, quantitative answer to this question. For each set $A \subseteq \mathbb{Z}^+$ and each nonnegative real number s, let

$$\zeta_A(s) = \sum_{n \in A} n^{-s}. \tag{2.1}$$

Note that $\zeta_{\mathbb{Z}^+}$ is precisely ζ, the Riemann zeta-function [24] (actually, Euler's original version [11] of the zeta-function, since we only consider $\zeta_A(s)$ for real s). The *zeta-dimension* of a set $A \subseteq \mathbb{Z}^+$ is then defined to be

$$\mathrm{Dim}_\zeta(A) = \inf\{s | \zeta_A(s) < \infty\}. \tag{2.2}$$

Since $\zeta_{\mathbb{Z}^+}(s) < \infty$ for all $s > 1$, we have

$$0 \leq \mathrm{Dim}_\zeta(A) \leq 1$$

for every set $A \subseteq \mathbb{Z}^+$. By the results cited in the preceding paragraph, the set of all positive integers and the set of all prime numbers each have zeta-dimension 1. Every finite set has zeta-dimension 0, because $\zeta_A(0)$ is the cardinality of A. It is easy to see that the set of nonnegative integer powers of 2 also has zeta-dimension 0. For a deeper example, Wirsing's $n^{O(\frac{1}{\ln \ln n})}$ upper bound on the number of perfect numbers not exceeding n [33] implies that the set of perfect numbers also has zeta-dimension 0.

The zeta-dimension of a set of positive integers can also lie strictly between 0 and 1. For example, if A is the set of all perfect squares, then $\zeta_A(s) = \zeta(2s)$, so $\mathrm{Dim}_\zeta(A) = \frac{1}{2}$. Similarly, the set of all perfect cubes has zeta-dimension $\frac{1}{3}$, etc. In fact, this argument can easily be extended to show that, for every real number $\alpha \in [0, 1]$, there exist sets $A \subseteq \mathbb{Z}^+$ such that $\mathrm{Dim}_\zeta(A) = \alpha$.

Intuitively, we regard zeta-dimension as a fractal dimension, analogous to Hausdorff dimension [14, 12] or (more aptly, as we shall see) packing dimension [30, 29, 12], on the space $\mathbb{Z}^+$ of positive integers. This intuition is supported by the fact that zeta-dimension has the following easily verified functional properties of a fractal dimension.

1. Monotonicity: $A \subseteq B$ implies $\mathrm{Dim}_\zeta(A) \leq \mathrm{Dim}_\zeta(B)$.

2. Stability: $\mathrm{Dim}_\zeta(A \cup B) = \max\{\mathrm{Dim}_\zeta(A), \mathrm{Dim}_\zeta(B)\}$.

3. Translation invariance: For each $k \in \mathbb{Z}^+$, $\mathrm{Dim}_\zeta(k + A) = \mathrm{Dim}_\zeta(A)$, where $k + A = \{k + n | n \in A\}$.

4. Expansion invariance: For each $k \in \mathbb{Z}^+$, $\mathrm{Dim}_\zeta(kA) = \mathrm{Dim}_\zeta(A)$, where $kA = \{kn | n \in A\}$.

Equation (2.1) can be written as a Dirichlet series

$$\zeta_A(s) = \sum_{n=1}^{\infty} f(n) n^{-s} \tag{2.3}$$

in which f is the characteristic function of A. In the terminology of analytic number theory, (2.2) then says that the zeta-dimension of A is the *abscissa of convergence* of the series (2.3) [17, 13, 2, 3]. In this sense, zeta-dimension was introduced in 1837 by Dirichlet [9]. The following useful characterization of zeta-dimension was proven in this more general setting in 1894.

Theorem 2.1 (entropy characterization of zeta-dimension – Cahen [5]; see also [16, 17, 13, 2, 3]). *For all $A \subseteq \mathbb{Z}^+$,*

$$\mathrm{Dim}_\zeta(A) = \limsup_{n\to\infty} \frac{\log |A \cap \{1,\dots,n\}|}{\log n}. \tag{2.4}$$

Example 2.2. The set C', consisting of all positive integers whose ternary expansions do not contain a 1, can be regarded as a discrete analog of the Cantor middle thirds set C, which consists of all real numbers in $[0, 1]$ who ternary expansions do not contain a 1. Theorem 2.1 implies immediately that C' has zeta-dimension $\frac{\log 2}{\log 3} \approx 0.6309$, which is exactly the classical fractal (Hausdorff or packing) dimension of C. We will see in section 4 that this is not a coincidence, but rather a special case of a general phenomenon.

By Theorem 2.1 and routine calculus, we have

$$\mathrm{Dim}_\zeta(A) = \limsup_{n\to\infty} \frac{\log |A \cap \{1,\dots,2^n\}|}{n} \tag{2.5}$$

and

$$\mathrm{Dim}_\zeta(A) = \limsup_{n\to\infty} \frac{\log |A \cap \{2^n,\dots,2^{n+1}-1\}|}{n} \tag{2.6}$$

for all $A \subseteq \mathbb{Z}^+$. The right-hand side of (2.6) has been called the (*channel*) *capacity* of A and the *entropy* (*rate*) of A [26, 18, 10, 6, 8, 27]. In particular, Staiger [27] (see also [15]) rediscovered (2.6) as a characterization of the entropy of A.

The following section shows how to extend zeta-dimension to the integer lattices $\mathbb{Z}^d$, for $d \geq 1$.

3 Zeta-Dimension in $\mathbb{Z}^d$

For each $\vec{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, where d is a positive integer, let $\|\vec{n}\|$ be the Euclidean distance from the origin to $\vec{n}$, i.e.,

$$\|\vec{n}\| = \sqrt{n_1^2 + \cdots + n_d^2}. \tag{3.1}$$

For each $A \subseteq \mathbb{Z}^d$, define the *A-zeta-function* $\zeta_A : [0,\infty) \to [0,\infty]$ by

$$\zeta_A(s) = \sum_{\vec{0}\neq\vec{n}\in A} \|\vec{n}\|^{-s} \tag{3.2}$$

for all $s \in [0,\infty)$, and define the *zeta-dimension* of A to be

$$\mathrm{Dim}_\zeta(A) = \inf\{s \mid \zeta_A(s) < \infty\}. \tag{3.3}$$

Note that, if $d = 1$ and $A \subseteq \mathbb{Z}^+$, then definitions (3.2) and (3.3) agree with definitions (2.1) and (2.2), respectively. The zeta-dimension that we have defined in $\mathbb{Z}^d$ is thus an extension of the one that was defined in $\mathbb{Z}^+$.

Observation 3.1. *For all $d \in \mathbb{Z}^+$ and $A \subseteq \mathbb{Z}^d$,*

$$0 \leq \mathrm{Dim}_\zeta(A) \leq d.$$

We next note that zeta-dimension has key properties of a fractal dimension in $\mathbb{Z}^d$. We state the invariance property a bit more generally than in section 2.

Definition. A function $f : \mathbb{Z}^d \to \mathbb{Z}^d$ is *bi-Lipschitz* if there exists $\alpha, \beta \in (0, \infty)$ such that, for all $\vec{m}, \vec{n} \in \mathbb{Z}^d$,

$$\alpha\|\vec{m} - \vec{n}\| \leq \|f(\vec{m}) - f(\vec{n})\| \leq \beta\|\vec{m} - \vec{n}\|.$$

Observation 3.2 (fractal properties of zeta-dimension). *Let $A, B \subseteq \mathbb{Z}^d$.*

1. *Monotonicity: $A \subseteq B$ implies $\mathrm{Dim}_\zeta(A) \leq \mathrm{Dim}_\zeta(B)$.*
2. *Stability: $\mathrm{Dim}_\zeta(A \cup B) = \max\{\mathrm{Dim}_\zeta(A), \mathrm{Dim}_\zeta(B)\}$.*
3. *Lipschitz invariance: If $f : \mathbb{Z}^d \to \mathbb{Z}^d$ is bi-Lipschitz, then $\mathrm{Dim}_\zeta(f(A)) = \mathrm{Dim}_\zeta(A)$.*

For $A \subseteq \mathbb{Z}^d$ and $I \subseteq [0, \infty)$, let

$$A_I = \{\vec{n} \in A \mid \|\vec{n}\| \in I\}.$$

Then the Dirichlet series

$$\zeta_A^D(s) = \sum_{n=1}^{\infty} |A_{[n,n+1)}| n^{-s} = \sum_{\vec{0} \neq \vec{n} \in A} \lfloor \|\vec{n}\| \rfloor^{-s}, \tag{3.4}$$

converges exactly when $\zeta_A(s)$ converges, so equation (3.3) says that $\mathrm{Dim}_\zeta(A)$ is the abscissa of convergence of this series. Cahen's 1894 characterization of this abscissa thus gives us the following extension of Theorem 2.1.

Theorem 3.3 (entropy characterization of zeta-dimension in $\mathbb{Z}^d$ – Cahen [5]). *For all $A \subseteq \mathbb{Z}^d$,*

$$\mathrm{Dim}_\zeta(A) = \limsup_{n \to \infty} \frac{\log |A_{[1,n]}|}{\log n}. \tag{3.5}$$

As in $\mathbb{Z}^+$, it follows immediately by routine calculus that

$$\mathrm{Dim}_\zeta(A) = \limsup_{n \to \infty} \frac{\log |A_{[1,2^n]}|}{n} \tag{3.6}$$

and

$$\mathrm{Dim}_\zeta(A) = \limsup_{n \to \infty} \frac{\log |A_{[2^n,2^{n+1})}|}{n} \tag{3.7}$$

for all $A \subseteq \mathbb{Z}^d$. Willson [31] has used (a quantity formally identical to) the right-hand side of (3.6) as a measure of the *growth-rate dimension* of a cellular automaton.

We next note that "subspaces" of $\mathbb{Z}^d$ have the "correct" zeta-dimensions.

Theorem 3.4. *If $\vec{m}_1, \ldots, \vec{m}_k \in \mathbb{Z}^d$ are linearly independent (as vectors in $\mathbb{R}^d$) and*

$$S = \{a_1\vec{m}_1 + \cdots + a_k\vec{m}_k \mid a_1, \ldots, a_k \in \mathbb{Z}\},$$

then $\mathrm{Dim}_\zeta(S) = k$.

By translation invariance, it follows that "hyperplanes" in $\mathbb{Z}^d$ also have the "correct" zeta-dimensions.

The Euclidean norm (3.1) is sometimes inconvenient for calculations. When desirable, the L^1 norm,

$$\|\vec{n}\|_1 = |n_1| + \cdots + |n_d|,$$

can be used in its place. That is, if we define the L^1 *A-zeta-function* $\zeta_A^{L^1}$ by

$$\zeta_A^{L^1}(s) = \sum_{\vec{0}\neq\vec{n}\in A} \|\vec{n}\|_1^{-s},$$

then

$$2^{-s}\zeta_A(s) \leq \zeta_A^{L^1}(s) \leq \zeta_A(s)$$

holds for all $s \in [0, \infty)$, so

$$\mathrm{Dim}_\zeta(A) = \inf\{s \mid \zeta_A^{L^1}(s) < \infty\}.$$

The entropy characterizations (3.5), (3.6), and (3.7) also hold with each set A_I replaced by the set

$$A_I^{L^1} = \{\vec{n} \in A \mid \|\vec{n}\|_1 \in I\}.$$

Example 3.5 (Pascal's triangle modulo 2)**.** Let

$$A = \{(m, n) \in \mathbb{N}^2 \mid \binom{m+n}{m} \equiv 1 \mod 2\}.$$

Then it is easy to see that $|A_{[1,2^n]}^{L^1}| = 3^n$ for all $n \in \mathbb{N}$, whence the L^1 version of (3.6) tells us that $\mathrm{Dim}_\zeta(A) = \log 3 \approx 1.5850$. This is exactly the fractal (Hausdorff or packing) dimension of the Sierpinski triangle that A so famously resembles [28]. This connection will be further illuminated in section 4.

In order to examine the zeta-dimensions of Cartesian products, we define the *lower zeta-dimension* of a set $A \subseteq \mathbb{Z}^+$ to be

$$\mathrm{dim}_\zeta(A) = \liminf_{n\to\infty} \frac{\log|A_{[1,n]}|}{\log n}. \tag{3.8}$$

By Theorem 3.3, $\mathrm{dim}_\zeta(A)$ is a sort of dual of $\mathrm{Dim}_\zeta(A)$. By routine calculus, we also have

$$\mathrm{dim}_\zeta(A) = \liminf_{n\to\infty} \frac{\log|A_{[1,2^n]}|}{n}, \tag{3.9}$$

i.e., the dual of equation (3.6) holds. Note, however, that the dual of equation (3.7) does *not* hold in general.

The following theorem is exactly analogous to a classical theorem on the Hausdorff and packing dimensions of Cartesian products [12].

Theorem 3.6. *For all $A \subseteq \mathbb{Z}^{d_1}$ and $B \subseteq \mathbb{Z}^{d_2}$,*

$$\begin{aligned}\mathrm{dim}_\zeta(A) + \mathrm{dim}_\zeta(B) &\leq \mathrm{dim}_\zeta(A \times B)\\ &\leq \mathrm{dim}_\zeta(A) + \mathrm{Dim}_\zeta(B)\\ &\leq \mathrm{Dim}_\zeta(A \times B)\\ &\leq \mathrm{Dim}_\zeta(A) + \mathrm{Dim}_\zeta(B).\end{aligned}$$

Although connectivity properties play an important role in classical fractal geometry, their role in discrete settings like $\mathbb{Z}^d$ will perforce be more limited. Nevertheless, we have the following. Given $d, r \in \mathbb{Z}^+$, and points $\vec{m}, \vec{n} \in \mathbb{Z}^d$, an *$r$-path* from $\vec{m}$ to $\vec{n}$ is a sequence $\pi = (\vec{p_0}, \ldots, \vec{p_l})$ of points $\vec{p_i} \in \mathbb{Z}^d$ such that $\vec{p_0} = \vec{m}$, $\vec{p_l} = \vec{n}$, and $\|\vec{p_i} - \vec{p_{i+1}}\| \leq r$ for all $0 \leq i < l$. Call a set $A \subseteq \mathbb{Z}^d$ *boundedly connected* if there exists $r \in \mathbb{Z}^+$ such that, for all $\vec{m}, \vec{n} \in A$, there is an r-path $\pi = (\vec{p_0}, \ldots, \vec{p_l})$ from $\vec{m}$ to $\vec{n}$ in which $\vec{p_i} \in A$ for all $0 \leq i \leq l$.

A result of classical fractal geometry says that any set of dimension less than 1 is totally disconnected. The following theorem is an analog of this for zeta-dimension.

Theorem 3.7. *Let $d \in \mathbb{Z}^+$ and $A \subseteq \mathbb{Z}^d$. If $\mathrm{Dim}_\zeta(A) < 1$, then no infinite subset of A is boundedly connected.*

The next section examines the relationships between zeta-dimension and classical fractal dimensions in greater detail.

4 Zeta-Dimension and Classical Fractal Dimension

The following result shows that the agreement between zeta-dimension and Hausdorff dimension noticed in Examples 2.2 and 3.5 are instances of a more general phenomenon: Given any discrete fractal with enough self similarity, its zeta-dimension is equal to the Hausdorff dimension of its classical version. Previous results along these lines were proven by Willson [31, 32], for the special case of sets that are obtained from additive cellular automata.

The following states what is meant by self-similarity precisely.

Definition. Let $c, d \in \mathbb{N}$, $F \subset \mathbb{N}^d$. F is a c-discrete self similar fractal, if there exists a function

$$S : \{1, 2, \cdots, c\}^d \to \{\mathrm{no}, R_0, R_1, R_2, R_3\}$$

such that $S(1, 1, \cdots, 1) = R_0$, and for every integer k and every $(i_1, \cdots, i_d) \in \{1, 2, \cdots, c\}^d$,

$$F \cap C^k_{i_1, i_2, \cdots, i_d} = \begin{cases} R_j(C^k_{1, \cdots, 1}) & \text{if } S(i_1, \cdots, i_d) = R_j,\\ \varnothing & \text{if } S(i_1, \cdots, i_d) = \mathrm{no}\end{cases}$$

where R_j ($j = 0, \cdots, 3$) is a rotation of angle $j\pi/2$, and

$$C^k_{i_1, i_2, \cdots, i_d} = [(i_1 - 1)c^k + 1, i_1 c^k] \times \cdots \times [(i_d - 1)c^k + 1, i_d c^k]$$

is a d-dimensional cube of side c.

There are many ways to generalize the above definition including statistical similarity, multiple patterns, fractal curves constructed from a generator [12], multiple contraction ratio (of the form $c_1, \cdots, c_n$ where $c_i | c_n$ for $i < n$). Also the preserved cube does not need to be $C^k_{1,\cdots,1}$, but can be any cube C, in which case the discrete fractal will grow in $\mathbb{Z}^d$ starting from C. It is easy to see that the following result still holds for those more general cases.

Given any c-discrete self similar fractal $F \subset \mathbb{N}^d$, we construct its continuous analogue $\mathbb{F} \subset [0, 1]^d$ recursively, via the following contraction $T : x \mapsto \frac{1}{c}x$. $\mathbb{F}_0 = [0, 1]$ and $\mathbb{F}_k = T^{(k)}(F \cap [1, c^k]^d)$, where $T^{(k)} = T\circ\cdots\circ T$, denotes k iterations of T. The fractal $\mathbb{F} = \lim_{k\to\infty} \mathbb{F}_k$ obtained by this construction is a self-similar continuous fractal with contraction ratio $1/c$. The following result shows that the zeta-dimension of the discrete fractal is equal to the Hausdorff dimension of the continuous one.

Theorem 4.1. *If $c, d, F, \mathbb{F}$ are as above, then*

$$\mathrm{Dim}_\zeta(F) = \dim_{\mathrm{H}}(\mathbb{F}).$$

The following result gives a relationship between zeta-dimension and dimension in the Baire space. We consider the Baire space $\mathbb{N}^\infty$ representing total functions from $\mathbb{N}$ to $\mathbb{N}$ in the obvious way. Given $w \in \mathbb{N}^*$, let $C_w = \{z \in \mathbb{N}^\infty | w \sqsubset z\}$. We define real : $\mathbb{N}^\infty \to [0, 1]$ by

$$\mathrm{real}(z) = \cfrac{1}{(z_0 + 1) + \cfrac{1}{(z_1 + 1) + \cdots}}.$$

The cylinder generated by w is the interval $\Delta(w) = \{x \in [0, 1] | x = \mathrm{real}(z), w \sqsubset z\}$.

A subprobability supermeasure on $\mathbb{N}^\infty$ is a function $p : \mathbb{N}^* \to [0, 1]$ such that $p(\lambda) \leq 1$ and for each $w \in \mathbb{N}^*$, $p(w) \geq \sum_n p(wn)$.

For each subprobability supermeasure p we can define a Hausdorff dimension and a packing dimension on $\mathbb{N}^\infty$, $\dim_p$ and Dim_p, using the metric ρ defined as $\rho(z, z') = p(w)$ for $w \in \mathbb{N}^*$ the longest common prefix of $z, z' \in \mathbb{N}^\infty$.

Gauss measure is defined on each $E \subseteq \mathbb{R}$ as

$$\gamma(E) = \frac{1}{\ln 2} \int_E \frac{dt}{1 + t}.$$

We will abuse notation and use $\gamma(w) = \gamma(\mathrm{real}(C_w))$ for each $w \in \mathbb{N}^*$. Notice that $\gamma(\lambda) = 1$ and therefore γ is a probability measure on $\mathbb{N}^\infty$.

Remark. Let μ denote the Lebesgue measure on $\mathbb{R}$, and let $\mu(w) = \mu(\mathrm{real}(C_w))$ for each $w \in \mathbb{N}^*$, then

$$\frac{\mu(w)}{2 \ln 2} \leq \gamma(w) \leq \frac{\mu(w)}{\ln 2},$$

so μ and γ give equivalent Hausdorff dimensions.

Define $F_A = \{f : \mathbb{N} \to \mathbb{N} | f(\mathbb{N}) \subseteq A \text{ and } \lim_{n\to\infty} f(n) = \infty\}$, for each $A \subseteq \mathbb{Z}^+$. The following result relates zeta-dimension to Gauss-dimension.

Theorem 4.2. $\mathrm{Dim}_\zeta(A) = 2 \cdot \dim_\gamma(F_A) = 2 \cdot \mathrm{Dim}_\gamma(F_A)$.

5 Zeta-Dimension and Algorithmic Information

The entropy characterization of zeta-dimension (Theorem 3.3) already indicates a strong connection between zeta-dimension and information theory. Here we explore further such connections. The first concerns the zeta-dimensions of sets of positive integers that are defined in terms of the digits, or strings of digits, that can appear in the base-k expansions of their elements. We write $\mathrm{rep}_k(n)$ for the base-k expansion ($k \geq 2$) of a positive integer n. Conversely, given a nonempty string $w \in \{0, 1, \cdots, k-1\}^*$ that does not begin with 0, we write $\mathrm{num}_k(w)$ for the positive integer whose base-k expansion is w.

A *prefix set* over an alphabet Σ is a set $B \subseteq \Sigma^*$ such that no element of B is a proper prefix of another element of B. An *instantaneous code* is a nonempty prefix set that does not contain the empty string.

Theorem 5.1. *Let $\Sigma = \{0, 1, \cdots, k-1\}$, where $k \geq 2$. Assume that $\varnothing \neq \Delta \subseteq \Sigma - \{0\}$ and that $B \subseteq \Sigma^*$ is a finite instantaneous code, and let*

$$A = \{n \in \mathbb{Z}^+ | \mathrm{rep}_k(n) \in \Delta B^*\}.$$

Then

$$\mathrm{Dim}_\zeta(A) = s^*,$$

where s^* is the unique solution of the equation

$$\sum_{w \in B} k^{-s^*|w|} = 1.$$

Corollary 5.2. *Let $\Sigma = \{0, 1, \cdots, k-1\}$, where $k \geq 2$. If $\Gamma \subseteq \Sigma$ and $\Gamma \nsubseteq \{0\}$ and*

$$A = \{n \in \mathbb{Z}^+ | \mathrm{rep}_k(n) \in \Gamma^*\},$$

then

$$\mathrm{Dim}_\zeta(A) = \frac{\ln|\Gamma|}{\ln k}.$$

Example 5.3. Corollary 5.2 gives a quantitative articulation of the "paradox of the missing digit"[13]. If A is the set of positive integers in whose decimal expansions some particular digit, such as 7, is missing, then a naive intuition might suggest that A contains "most" integers, but A has long been known to be small in the sense that the sum of the reciprocals of its elements is finite (i.e., $\zeta_A(1) < \infty$). In fact, Corollary 5.2 says that $\mathrm{Dim}_\zeta(A) = \frac{\ln 9}{\ln 10} \approx 0.9542$, a quantity somewhat smaller than, say, the zeta-dimension of the set of prime numbers.

The main connection between zeta-dimension and *algorithmic* information theory is a theorem of Staiger [27] relating entropy to Kolmogorov complexity. To state Staiger's theorem in our present framework, we define the *Kolmogorov complexity* $K(\vec{n})$ of a point $\vec{n} \in \mathbb{Z}^d$ to be the length of a shortest program $\pi \in \{0, 1\}^*$ such that, when a fixed universal self-delimiting Turing machine U is run with (π, d) as its input, U outputs $\vec{n}$ (actually, some straightforward encoding of $\vec{n}$ as a binary string) and halts after finitely many computation steps. Detailed discussions of Kolmogorov complexity's definition, fundamental properties, history, significance, and applications appear in the definitive textbook by Li and Vitanyi [19]. As we have already noted, $K(\vec{n})$ is a measure of the *algorithmic information content* of $\vec{n}$.

For $\vec{0} \neq \vec{n} \in \mathbb{Z}^d$, we write $l(\|\vec{n}\|)$ for the length of the standard binary expansion (no leading zeroes) of the positive integer $\lfloor \|\vec{n}\| \rfloor$.

If $f : \mathbb{Z}^d \to [0, \infty)$ and $A \subseteq \mathbb{Z}^d$, then the *limit superior* of f on A is

$$\limsup_{\vec{n} \in A} f(\vec{n}) = \lim_{k \to \infty} \sup f(A_{[k,\infty]}).$$

Note that this is 0 if A is finite.

Theorem 5.4 (Kolmogorov [34], Staiger [27]). *For every $A \subseteq \mathbb{Z}^d$,*

$$\mathrm{Dim}_\zeta(A) \leq \limsup_{\vec{n} \in A} \frac{\mathrm{K}(\vec{n})}{l(\|\vec{n}\|)},$$

with equality if A or its complement is computably enumerable.

In the case where $d = 1$ and $A \subseteq \mathbb{Z}^+$, Theorem 5.4 says that, if A is Σ^0_1 or Π^0_1, then

$$\mathrm{Dim}_\zeta(A) = \limsup_{n \in A} \frac{\mathrm{K}(n)}{l(n)},$$

where $l(n)$ is the length of the binary representation of A. Kolmogorov [34] proved this for Σ^0_1 sets, and Staiger [27] proved it for Π^0_1 sets. The extension to $A \subseteq \mathbb{Z}^d$ for arbitrary $d \in \mathbb{Z}^+$ is routine.

As Staiger has noted, Theorem 5.4 cannot be extended to Δ^0_2 sets, because an oracle for the halting problem can easily be used to decide a set $B \subseteq \mathbb{Z}^+$ such that, for each $k \in \mathbb{Z}^+$, $B_{[2^k, 2^{k+1}]}$ contains exactly one integer n, and this n also satisfies $\mathrm{K}(n) \geq k$. Such a set B is a Δ^0_2 set satisfying $\mathrm{Dim}_\zeta(B) = 0 < 1 = \limsup_{n \in B} \frac{\mathrm{K}(n)}{l(n)}$.

Classical Hausdorff and packing dimensions were recently characterized in terms of gales, which are betting strategies with a parameter s that quantifies how favorable the payoffs are [20, 4]. These characterizations have played a central role in many recent studies of effective fractal dimensions in algorithmic information theory and computational complexity theory [22]. We show here that zeta-dimension also admits such a characterization.

Briefly, given $s \in [0, \infty)$, an *s-gale* is a function $d : \{0,1\}^* \to [0, \infty)$ satisfying $d(w) = 2^{-s}[d(w0) + d(w1)]$ for all $w \in \{0,1\}^*$. For purposes of this paper, an s-gale d *succeeds* on a positive integer n if $d(w) \geq 1$, where w is the standard binary representation of n.

Theorem 5.5 (gale characterization of zeta-dimension). *For all $A \subseteq \mathbb{Z}^+$,*

$$\mathrm{Dim}_\zeta(A) = \inf\{s \mid \text{there is an } s\text{-gale } d \text{ that succeeds on every element of } A\}.$$

Our last result is a theorem on the zeta-dimensions of pointwise sums and products of sets of positive integers. For $A, B \subseteq \mathbb{Z}^+$, we use the notations

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\},$$
$$A * B = \{ab \mid a \in A \text{ and } b \in B\}.$$

The first equality in the following theorem is due to Staiger [27].

Theorem 5.6. *If $A, B \subseteq \mathbb{Z}^+$ are nonempty, then*

$$\mathrm{Dim}_\zeta(A * B) = \max\{\mathrm{Dim}_\zeta(A), \mathrm{Dim}_\zeta(B)\} \leq \mathrm{Dim}_\zeta(A + B) \leq \mathrm{Dim}_\zeta(A) + \mathrm{Dim}_\zeta(B),$$

and the inequalities are tight in the strong sense that, for all $\alpha, \beta, \gamma \in [0,1]$ with $\max\{\alpha, \beta\} \leq \gamma \leq \alpha + \beta$, there exist $A, B \subseteq \mathbb{Z}^+$ with $\mathrm{Dim}_\zeta(A) = \alpha$, $\mathrm{Dim}_\zeta(B) = \beta$, and $\mathrm{Dim}_\zeta(A + B) = \gamma$.

We close with a question concerning circuit definability of sets of natural numbers, a notion introduced recently by McKenzie and Wagner [23]. Briefly, a McKenzie-Wagner *circuit* is a combinational circuit (finite directed acyclic graph) in which the inputs are singleton sets of natural numbers, and each gate is of one of five types. Gates of type $\cup$, $\cap$, +, and $*$ have indegree 2 and compute set union, set intersection, pointwise sum, and pointwise product, respectively. Gates of type $^-$ have indegree 1 and compute set complement. Each such circuit *defines* the set of natural numbers computed at its designated output gate in the obvious way. The fact that 0 is a natural number is crucial in this model. Interesting sets that are known to be definable in this model include the set of primes, the set of powers of a given prime, and the set of counterexamples to Goldbach's conjecture. Is there a zero-one law, according to which every set definable by a McKenzie-Wagner circuit has zeta-dimension 0 or 1? Such a law would explain the fact that the set of perfect squares is not known to be definable by such circuits. Theorem 5.6 suggests that a zero-one law, if true, will not be proven by a trivial induction on circuits.

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A Appendix – Zeta-Dimension in $\mathbb{Z}^d$

Proof of Theorem 3.4. Assume the hypothesis. By standard results in the geometry of numbers, there exist constants $\alpha, \beta \in (0, \infty)$ such that, for all $n \in \mathbb{Z}^+$,

$$\alpha n^k \leq |S_{[1,n]}| \leq \beta n^k.$$

It follows by Theorem 3.3 that $\mathrm{Dim}_\zeta(S) = k$. □

Proof of Theorem 3.6. The following is easy to show.

Claim. *Let $A \subseteq \mathbb{Z}^{d_1}$, $B \subseteq \mathbb{Z}^{d_2}$ and $n \in \mathbb{N}$, then*

$$A_{[1,n]} \times B_{[1,n]} \subseteq (A \times B)_{[1,2n]} \subseteq A_{[1,2n]} \times B_{[1,2n]}$$

i.e.

$$|A_{[1,n]}| \cdot |B_{[1,n]}| \leq |(A \times B)_{[1,2n]}| \leq |A_{[1,2n]}| \cdot |B_{[1,2n]}|.$$

Let us prove the first inequality.

$$\begin{aligned}
\dim_\zeta(A) + \dim_\zeta(B) &= \liminf_{n\to\infty} \frac{\log |A_{[1,n]}|}{\log n} + \liminf_{n\to\infty} \frac{\log |A_{[1,n]}|}{\log n} \\
&\leq \liminf_{n\to\infty} \frac{\log(|A_{[1,n]}| \cdot |B_{[1,n]}|)}{\log n} \\
&\leq \liminf_{n\to\infty} \frac{\log(|(A \times B)_{[1,2n]}|)}{\log n} \\
&\leq \liminf_{n\to\infty} \frac{\log(|(A \times B)_{[1,2n]}|)}{\log 2n} = \dim_\zeta(A \times B)
\end{aligned}$$

For the second inequality we have

$$\begin{aligned}
\dim_\zeta(A \times B) &= \liminf_{n\to\infty} \frac{\log |(A \times B)_{[1,n]}|}{\log n} \\
&\leq \liminf_{n\to\infty} \frac{\log(|A_{[1,n]}| \cdot |B_{[1,n]}|)}{\log n} \\
&= \liminf_{n\to\infty} \frac{\log |A_{[1,n]}| + \log |B_{[1,n]}|}{\log n} \\
&\leq \liminf_{n\to\infty} \frac{\log |A_{[1,n]}|}{\log n} + \limsup_{n\to\infty} \frac{\log |B_{[1,n]}|}{\log n} = \dim_\zeta(A) + \mathrm{Dim}_\zeta(B)
\end{aligned}$$

For the third inequality we have

$$\begin{aligned}
\dim_\zeta(A) + \mathrm{Dim}_\zeta(B) &= \liminf_{n\to\infty} \frac{\log |A_{[1,n]}|}{\log n} + \limsup_{n\to\infty} \frac{\log |B_{[1,n]}|}{\log n} \\
&\le \limsup_{n\to\infty} \frac{\log |A_{[1,n]}| + \log |B_{[1,n]}|}{\log n} \\
&= \limsup_{n\to\infty} \frac{\log(|A_{[1,n]}| \cdot |B_{[1,n]}|)}{\log n} \\
&\le \limsup_{n\to\infty} \frac{\log(|(A\times B)_{[1,2n]}|)}{\log n} \\
&\le \limsup_{n\to\infty} \frac{\log(|(A\times B)_{[1,2n]}|)}{\log 2n} = \mathrm{Dim}_\zeta(A\times B)
\end{aligned}$$

For the last inequality we have

$$\begin{aligned}
\dim_\zeta(A\times B) &= \limsup_{n\to\infty} \frac{\log(|(A\times B)_{[1,n]}|)}{\log n} \\
&\le \limsup_{n\to\infty} \frac{\log |A_{[1,n]}|}{\log n} + \frac{\log |B_{[1,n]}|}{\log n} \\
&\le \limsup_{n\to\infty} \frac{\log |A_{[1,n]}|}{\log n} + \limsup_{n\to\infty} \frac{\log |B_{[1,n]}|}{\log n} = \mathrm{Dim}_\zeta(A) + \mathrm{Dim}_\zeta(B)
\end{aligned}$$

□

Proof of Theorem 3.7. Let $A \subseteq \mathbb{Z}^d$, and let C be an infinite, boundedly connected subset of A. It su□ces to prove that $\mathrm{Dim}_\zeta(A) \ge 1$.

Write $C = \{\vec{n_k} \mid k \in \mathbb{N}\}$. Since C is boundedly connected, there is, for each $k \in \mathbb{N}$, an r-path π_k from $\vec{n_k}$ to $\vec{n_{k+1}}$, all of whose points are in C. Inserting those paths into the list $\vec{n_0}, \vec{n_1}, \ldots$, we get an expanded list $\vec{m_0}, \vec{m_1}, \ldots$ of points in C such that (i) every point of C appears in the list $\vec{m_0}, \vec{m_1}, \ldots$; and (ii) for all $k \in \mathbb{N}$, $\|\vec{m_k}, \vec{m_{k+1}}\| \le r$. If we now delete from the list $\vec{m_0}, \vec{m_1}, \ldots$ each $\vec{m_k}$ that has appeared earlier in the list, then we obtain an enumeration $\vec{p_0}, \vec{p_1}, \ldots$ of C in which there is no repetition and

$$\|\vec{p_k}\| \le \|\vec{p_0}\| + kr$$

holds for all $k \in \mathbb{N}$. It follows that

$$\begin{aligned}
\zeta_A(1) &\ge \zeta_C(1) \\
&= \sum_{k=0}^{\infty} \|\vec{p_k}\|^{-1} \\
&\ge \sum_{k=0}^{\infty} \frac{1}{\|\vec{p_0}\| + kr} \\
&= \infty,
\end{aligned}$$

whence $\mathrm{Dim}_\zeta(A) \ge 1$. □

B Appendix – Zeta-Dimension and Classical Fractal Dimension

Proof of Theorem 4.1. Consider $F_k = [1, c^k]^d$ and let

$$B(F_k) = \frac{\log |F_k|}{k \log c} \quad \text{and} \quad B(F) = \lim_{k\to\infty} B(F_k).$$

Claim. $|F_k| = |S^{-1}(\{R_0, \cdots, R_3\})|^k$.

We prove the claim by induction. The claim is true for $k = 1$; let $k \in \mathbb{N}$, we have

$$|F_k| = |F_{k-1}| \cdot |S^{-1}(\{R_0, \cdots, R_3\})| = |S^{-1}(\{R_0, \cdots, R_3\})|^k.$$

This proves the claim.

Let $Y = |S^{-1}(\{R_0, \cdots, R_3\})|$. By the claim,

$$B(F) = \lim_{k\to\infty} B(F_k) = \lim_{k\to\infty} \frac{\log |F_k|}{k \log c} = \frac{\log Y}{\log c}.$$

Claim. $\mathrm{Dim}_\zeta(F) = B(F)$.

To prove the claim consider $D_k = F_{k+1} - F_k$. We have $|D_k| = Y^k(Y - 1)$. For a tuple $(m_1, \cdots, m_d) \in D_k$ we have

$$dc^k \le m_1 + \cdots + m_d \le dc^{k+1}$$

thus

$$d^{-s}c^{-s(k+1)} \le (m_1 + \cdots + m_d)^{-s} \le d^{-s}c^{-sk}$$

i.e.

$$|D_k| d^{-s}c^{-s(k+1)} \le \zeta_{D_k}(s) \le |D_k| d^{-s}c^{-sk}$$

therefore

$$Y^k(Y-1)d^{-s}c^{-s(k+1)} \le \zeta_{D_k}(s) \le Y^k(Y-1)d^{-s}c^{-sk}$$

thus

$$a\sum_{k\ge1}(Yc^{-s}) \le \mathrm{Dim}_\zeta(F) \le b\sum_{k\ge1}(Yc^{-s})$$

where a, b are constants. The convergence radius of the upper sum gives the zeta-dimension of F, i.e. is solution of the equation $Yc^{-s} = 1$, thus $s = \log Y/\log c$, which proves the claim.

Claim. $\dim_{\mathrm{H}}(\mathbb{F}) = B(F)$.

The box dimension of $\mathbb{F}$ is given by

$$\dim_{\mathrm{B}}(\mathbb{F}) = \lim_{k\to\infty} \frac{\log N_{c^{-k}}(\mathbb{F})}{k \log c}$$

where $N_{c^{-k}}$ is the number of d-mesh cubes of side c^{-k} of the form

$$M^k_{m_1,\cdots,m_d} = [m_1c^{-k}, (m_1+1)c^{-k}] \times \cdots [m_dc^{-k}, (m_d+1)c^{-k}], \quad \text{where} \quad m_i \in \mathbb{N}$$

required to cover $\mathbb{F}$.

Since $\mathbb{F} \subset \mathbb{F}_k$ we have $N_{c^{-k}}(\mathbb{F}) \leq N_{c^{-k}}(\mathbb{F}_k)$. Moreover the number of mesh cubes $M^k_{m_1,\cdots,m_d}$ required to cover $\mathbb{F}_k$ is equal to the number required to cover $\mathbb{F}_{k+j}$ for any integer j, because $\mathbb{F}_k \cap M^k_{m_1,\cdots,m_d} \neq \varnothing$ implies $\mathbb{F}_{k+j} \cap M^k_{m_1,\cdots,m_d} \neq \varnothing$ by construction. Thus $N_{c^{-k}}(\mathbb{F}) \geq N_{c^{-k}}(\mathbb{F}_k)$. Moreover by construction, $N_{c^{-k}}(\mathbb{F}_k) = |F_k|$. Therefore

$$\dim_{\mathrm{B}}(\mathbb{F}) = \lim_{k\to\infty} \frac{\log N_{c^{-k}}(\mathbb{F})}{k \log c} = \lim_{k\to\infty} \frac{\log N_{c^{-k}}(\mathbb{F}_k)}{k \log c} = \lim_{k\to\infty} \frac{|F_k|}{k \log c} = \frac{|Y|}{\log c}.$$

Since box dimension coincides with Hausdorff dimension on self similar continuous fractals, this ends the proof. □

Proof of Theorem 4.2. Let $s > \mathrm{Dim}_\zeta$, $\epsilon > 0$, and $C = \sum_{n\in A}(n+1)^{-s}$. Consider the following $(s/2+\epsilon)$-γ-supergale d, where $d(wn) = d(w)\frac{(n+1)^{2\epsilon}}{4C}$ for $n \in A$. For each $f \in F_A$, there is an m_0 such that $f(m)^{2\epsilon} > 8C$ for each $m \geq m_0$. Therefore, if $|w| = m$, $d(wf(m)) > 2d(w)$ and $F_A \subseteq S^\infty_{\mathrm{str}}[d]$.

For the other direction, let $t > \dim_\gamma(F_A)$ and let d be a t-gale such that $F_A \subseteq S^\infty[d]$. Then the supremum over all $w \in A^*$ of $\inf_{n\in A, n>|w|} d(wn)/d(w)$ is greater that 1 (otherwise we can construct f in $F_A - S^\infty[d]$). Thus $\sum_{n\in A}(n+1)^{-2t} < \infty$. □

C Appendix – Zeta-Dimension and Algorithmic Information

Proof for Theorem 5.1. Assume the hypothesis. For each $s \in [0,\infty)$, $a \in \Delta$, and $0 \leq t \in \mathbb{Z}$, let

$$\beta_s = \sum_{w\in B} k^{-s|w|}$$

and

$$g(s,a,t) = \sum_{(w_1,\cdots,w_t)\in B^t} \mathrm{num}_k(aw_1\cdots w_t)^{-s}.$$

Also, for each $\vec{w} = (w_1,\cdots,w_t) \in B^t$, write

$$l(\vec{w}) = \sum_{i=1}^{t} |w_i|.$$

Then, for all such s, a, and t, we have

$$\begin{aligned} g(s,a,t) &\leq \sum_{\vec{w}\in B^t} \mathrm{num}_k(a0^{l(\vec{w})})^{-s} \\ &= \sum_{\vec{w}\in B^t} (ak^{l(\vec{w})})^{-s} \\ &= a^{-s} \sum_{\vec{w}\in B^t} \prod_{i=1}^{t} k^{-s|w_i|} \\ &= a^{-s}\beta_s^t \end{aligned}$$

and

$$\begin{aligned} g(s,a,t) &\geq \sum_{\vec{w}\in B^t} \text{num}_k(a(k-1)^{l(\vec{w})})^{-s} \\ &\geq \sum_{\vec{w}\in B^t} ((a+1)k^{l(\vec{w})})^{-s} \\ &= (a+1)^{-s} \sum_{\vec{w}\in B^t} \prod_{i=1}^{t} k^{-s|w_i|} \\ &= (a+1)^{-s}\beta_s^t. \end{aligned}$$

That is, for all $s \in [0,\infty)$, $a \in \Delta$, and $0 \leq t \in \mathbb{Z}$,

$$(a+1)^{-s}\beta_s^t \leq g(s,a,t) \leq a^{-s}\beta_s^t. \tag{C.1}$$

Since B is an instantaneous code, we have

$$\zeta_A(s) = \sum_{a\in\Delta}\sum_{t=0}^{\infty} g(s,a,t) \tag{C.2}$$

for all $s \in [0,\infty)$. Putting (C.1) and (C.2) together gives

$$\sum_{a\in\Delta}(a+1)^{-s}\sum_{t=0}^{\infty}\beta_s^t \leq \zeta_A(s) \leq \sum_{a\in\Delta} a^{-s}\sum_{t=0}^{\infty}\beta_s^t$$

for all $s \in [0,\infty)$. By our choice of s^*, then,

$$s > s^* \Rightarrow \beta_s < 1 \Rightarrow \zeta_A(s) < \infty$$

and

$$s \leq s^* \Rightarrow \beta_s \geq 1 \Rightarrow \zeta_A(s) = \infty.$$

Thus $\text{Dim}_\zeta(A) = s^*$. □

Proof of Corollary 5.2. Apply Theorem 5.1 with $\Delta = \Gamma - \{0\}$ and $B = \Gamma$. □

Lemma C.1 (Kraft's inequality). *Let $s > 0$. Let d be an s-supergale. Then $\left|S^1[d] \cap \{0,1\}^k\right| \leq 2^{sk}d(\lambda)$ for all $k \in \mathbb{N}$.*

Proof. Let $A = S^1[d] \cap \{0,1\}^k$. Since d is an s-supergale, we have for every $w \in A$, $d(w) \geq 1$. By the definition of supergale, we know that

$$\sum_{w\in\{0,1\}^k} d(w) \leq 2^{sk}d(\lambda).$$

Therefore

$$\begin{aligned} |A|\cdot 1 &\leq \sum_{w\in A} d(w) \\ &\leq \sum_{w\in\{0,1\}^k} d(w) \\ &\leq 2^{sk}d(\lambda). \end{aligned}$$

□

Proof of Theorem 5.5. Let $s > 0$. First, we show that for any s-supergale d,

$$\dim_\zeta(\mathrm{bnum}(S^1[d] \cap 1\{0,1\}^*)) \leq s.$$

Let $A = \mathrm{bnum}(S^1[d] \cap 1\{0,1\}^*)$. Let $\epsilon > 0$.

$$\begin{aligned}
\zeta_A(s+\epsilon) &= \sum_{x \in A} \frac{1}{x^{s+\epsilon}} \leq \sum_{w \in S^1[d] \cap 1\{0,1\}^*} \frac{1}{\mathrm{bnum}(w)^{s+\epsilon}} \\
&\leq s^{s+\epsilon} \sum_{w \in S^1[d] \cap 1\{0,1\}^*} \frac{1}{2^{(s+\epsilon)|w|}} \\
&\leq s^{s+\epsilon} \sum_{k=0}^{\infty} \sum_{\substack{w \in S^1[d] \\ |w|=k}} \frac{1}{2^{(s+\epsilon)|w|}} \\
&= s^{s+\epsilon} \sum_{k=0}^{\infty} \left| S^1[d] \cap \{0,1\}^k \right| \frac{1}{2^{(s+\epsilon)k}} \\
&\leq^{\text{by Lemma C.1}} s^{s+\epsilon} \sum_{k=0}^{\infty} 2^{sk} d(\lambda) \frac{1}{2^{(s+\epsilon)k}} \\
&= s^{s+\epsilon} d(\lambda) \sum_{k=0}^{\infty} \frac{1}{2^{\epsilon k}} < \infty.
\end{aligned}$$

Since ϵ is arbitrary, $\dim_\zeta(A) \leq s$.

Now we prove that if $\dim_\zeta(A) < s$, then there exists an s-supergale d such that $A \subseteq \mathrm{bnum}(S^1[d])$.

Since $\dim_\zeta(A) < s$, for some $\epsilon > 0$,

$$\sum_{k=0}^{\infty} \frac{|A_{=k}|}{2^{(s-\epsilon)k}} = \sum_{k=0}^{\infty} \sum_{\substack{w \in \{0,1\}^k \\ \mathrm{bnum}(w) \in A_{=k}}} \frac{1}{2^{(s-\epsilon)k}} \leq \sum_{x \in A} \frac{1}{x^{s-\epsilon}} = \zeta_A(s-\epsilon) < \infty.$$

Thus there exists $n_0 \in \mathbb{Z}^+$, such that for all $k > n_0$,

$$\frac{|A_{=k}|}{2^{(s-\epsilon)k}} < 1.$$

Let

$$C_0 = \max\left\{1, \frac{|A_{=1}|}{2^{(s-\epsilon)1}}, \frac{|A_{=2}|}{2^{(s-\epsilon)2}}, \ldots, \frac{|A_{=n_0}|}{2^{(s-\epsilon)n_0}}\right\}.$$

Let

$$C_1 = \max_{n \in \mathbb{Z}^+} \left\{ \frac{n^2}{2^{\epsilon n}} \right\}.$$

Since $\frac{n^2}{2^{\epsilon n}}$ is eventually monotone decreasing, $C_1 < \infty$ exists.

We construct an s-supergale as follows.

For every $k \in \mathbb{Z}^+$, let $d_k : \{0,1\}^* \to [0,\infty)$ be defined by the following recursion. And without loss of generality, for our convenience, we assume that $|A_{=k}| \geq 1$ for all $k \in \mathbb{Z}^+$.

$$d_k(w) = \begin{cases} \frac{2^k}{|A_{=k}|}, & |w| = k \text{ and } w \in A_{=k}, \\ 0, & |w| = k \text{ and } w \notin A_{=k}, \\ \frac{d_k(w0)+d_k(w1)}{2}, & |w| < k, \\ d_k(w[0..k-1]), & |w| > k. \end{cases}$$

Let

$$d(w) = C_0 C_1 2^{(s-1)|w|} \sum_{k=1}^{\infty} \frac{1}{k^2} d_k(w).$$

It is easy to verify that d_k's are martingales and d is an s-supergale.

Now let $x \in A$ and assume $x = \text{bnum}(w)$ and $|w| = n \in \mathbb{Z}^+$.

$$\begin{aligned} d(w) &= C_0 C_1 2^{(s-1)|w|} \sum_{k=0}^{\infty} \frac{1}{k^2} d_k(w) \geq C_0 C_1 2^{(s-1)n} \frac{1}{n^2} d_n(w) \\ &= C_0 C_1 2^{(s-1)n} \frac{1}{n^2} \frac{2^n}{|A_{=n}|} \geq C_0 C_1 2^{(s-1)n} \frac{1}{n^2} \frac{2^n}{C_0 2^{(s-\epsilon)n}} \\ &= C_1 \frac{2^{\epsilon n}}{n^2} \geq 1. \end{aligned}$$

Therefore, $w \in S^1[d]$, i.e., $x = \text{bnum}(w) \in \text{bnum}(S^1[d])$. □

Theorem C.2. *Let $\alpha, \beta, \gamma \in [0,1]$ and $\alpha < \beta \leq \gamma \leq \min\{1, \alpha + \beta\}$, then there exist $A, B \subseteq \mathbb{Z}^+$ such that $\text{Dim}_\zeta(A) = \alpha$, $\text{Dim}_\zeta(B) = \beta$ and $\text{Dim}_\zeta(A+B) = \gamma$.*

Proof. Let

$$A_1 = \{x \in \mathbb{Z}^+ \mid x \geq 2^{|\text{rep}_2(x)|-1} \text{ and } x < 2^{|\text{rep}_2(x)|-1} + \left\lceil 2^{\alpha|\text{rep}_2(x)|} \right\rceil\}$$

Let

$$B_1 = \{x \in \mathbb{Z}^+ \mid x \geq 2^{|\text{rep}_2(x)|-1} \text{ and } x < 2^{|\text{rep}_2(x)|-1} + \left\lceil 2^{\beta|\text{rep}_2(x)|} \right\rceil\}$$

and

$$B_2 = \{x \in \mathbb{Z}^+ \mid x = 2^{|\text{rep}_2(x)|-1} + k \left\lfloor 2^{\alpha|\text{rep}_2(x)|} \right\rfloor, 0 \leq k < \left\lceil 2^{(\gamma-\alpha)|\text{rep}_2(x)|} \right\rceil\}$$

Let $T : \mathbb{Z}^+ \to \mathbb{Z}^+$ be such that $T(1) = 1$ and $T(n+1) = 2^{T(n)}$.

Let

$$B = (B_1 \cup B_2) \cap \{x \mid |x| = T(n) \text{ for some } n \in \mathbb{Z}^+\}$$

and

$$A = A_1 \cap \{x \mid |x| = T(n) \text{ for some } n \in \mathbb{Z}^+\}.$$

Let $C = A + B$. Let $n = T(k)$ for some $k \in \mathbb{Z}^+$. Then

$$\left\{ x \,\middle|\, x \geq 2^{n-1} + 2^{n-1} \text{ and } x < 2^{n-1} + \lceil 2^{\alpha n} \rceil + 2^{n-1} + \lfloor 2^{\alpha n} \rfloor \left\lceil 2^{(\gamma-\alpha)n} - 1 \right\rceil \right\} = C_{=n+1},$$

i.e.,

$$\left\{ x \,\middle|\, x \geq 2^n \text{ and } x < 2^n + \lceil 2^{\alpha n} \rceil + \lfloor 2^{\alpha n} \rfloor \left\lceil 2^{(\gamma-\alpha)n} - 1 \right\rceil \right\} = C_{=n+1},$$

and

$$C_{=n} \subseteq B_{=n} + A_{\leq \log n}.$$

It is easy to verify that

$$|C_{=n}| \leq |B_{=n} + A_{\leq \log n}| \leq n \left\lceil 2^{(\gamma-\alpha)n} \right\rceil$$

and

$$|C_{=n+1}| = \lceil 2^{\alpha n} \rceil + \lfloor 2^{\alpha n} \rfloor \left\lceil 2^{(\gamma-\alpha)n} - 1 \right\rceil,$$

i.e.,

$$2^{\gamma n} - 2^{(\gamma-\alpha)n} \leq |C_{=n+1}| \leq 2 \cdot 2^{\gamma n}.$$

For $n \neq T(k)$ and $n \neq T(k) + 1$ for some $k \in \mathbb{Z}^+$, it is easy to verify that $C_{=n} = \varnothing$. It is now clear that the entropy rate of C

$$\mathrm{H}_C = \limsup_{n\to\infty} \frac{\log |C_{=n+1}|}{n+1} = \limsup_{k\to\infty} \frac{\log |C_{=T(k)+1}|}{T(k)+1} = \gamma,$$

i.e, $\mathrm{Dim}_\zeta(C) = \gamma$. Similarly, it is easy to verify that $\mathrm{Dim}_\zeta(A) = \alpha$ and $\mathrm{Dim}_\zeta(B) = \beta$. □

Proof of Theorem 5.6. Let $\alpha = \mathrm{Dim}_\zeta(A)$, $\beta = \mathrm{Dim}_\zeta(B)$ and without loss of generality assume $\alpha \geq \beta$. By Theorem C.2 and Staiger's proof that $\mathrm{Dim}_\zeta(A * B) = \max\{\mathrm{Dim}_\zeta(A), \mathrm{Dim}_\zeta(B)\}$ [27], it su□ces to show that

$$\max\{\alpha, \beta\} \leq \mathrm{Dim}_\zeta(A + B)$$

and

$$\mathrm{Dim}_\zeta(A + B) \leq \alpha + \beta.$$

For the first inequality, let $b = \min B$. Then it is easy to see that $\mathrm{Dim}_\zeta(A + B) \geq \mathrm{Dim}_\zeta(A + \{b\})$. Since zeta-dimension is invariant under translation, $\mathrm{Dim}_\zeta(A + \{b\}) = \mathrm{Dim}_\zeta(A) = \alpha = \max\{\alpha, \beta\}$.

For the second inequality, let $\epsilon > 0$. Since $\mathrm{Dim}_\zeta(A) = \alpha$ and $\mathrm{Dim}_\zeta(B) = \beta$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|A_{=n}| \leq 2^{(\alpha+\epsilon)n}$ and $|B_{=n}| \leq 2^{(\beta+\epsilon)n}$. Let

$$C = \max\{\sum_{n=1}^{n_0-1} |A_{=n}|, \sum_{n=1}^{n_0-1} |B_{=n}|\}.$$

It is clear that

$$\begin{aligned}
|(A + B)_{=n}| &\leq (|A_{=n}| + |A_{=n-1}|) \sum_{k=1}^{n} |B_{=n}| + (|B_{=n}| + |B_{=n-1}|) \sum_{k=1}^{n} |A_{=n}| \\
&\leq (|A_{=n}| + |A_{=n-1}|) \sum_{k=n_0}^{n} |B_{=n}| + (|B_{=n}| + |B_{=n-1}|) \sum_{k=n_0}^{n} |A_{=n}| \\
&\quad + C(|A_{=n}| + |A_{=n-1}|) + C(|B_{=n}| + |B_{=n-1}|) \\
&\leq (1 + 2^{\alpha+\epsilon}) 2^{(\alpha+\epsilon)n} \frac{2^{(\beta+\epsilon)n_0} (2^{(\beta+\epsilon)(n-n_0)} - 1)}{2^{\beta+\epsilon}} \\
&\quad + (1 + 2^{\beta+\epsilon}) 2^{(\beta+\epsilon)n} \frac{2^{(\alpha+\epsilon)n_0} (2^{(\alpha+\epsilon)(n-n_0)} - 1)}{2^{\alpha+\epsilon}} \\
&\quad + C(1 + 2^{\alpha+\epsilon}) 2^{(\alpha+\epsilon)n} + C(1 + 2^{\beta+\epsilon}) 2^{(\beta+\epsilon)n}.
\end{aligned}$$

Let

$$C' = \max\{C(1 + 2^{\alpha+\epsilon}2^{(\beta+\epsilon)n_0}), C(1 + 2^{\beta+\epsilon}2^{(\alpha+\epsilon)n_0})\}.$$

Then for all $n \geq n_0$

$$|(A + B)_{=n}| \leq C' 2^{(\alpha+\beta+2\epsilon)n}.$$

By the entropy characterization of zeta-dimension, it is clear that $\text{Dim}_\zeta(A + B) \leq \alpha + \beta$. □