PROBLEMS AND NOTES FROM LECTURE 2

Recall some definitions.

Definition 1. A metric space X is a set together with a metric d_X. A metric d_X is a binary map from $X \times X$ to $\mathbb{R}$, that is:

$$d_X : X \times X \to \mathbb{R}$$

satisfying three properties:

(1) $\forall x, y \in X$ we have $d_X(x, y) \geq 0$ and $d_X(x, y) = 0$ if and only if $x = y$
(2) $\forall x, y \in X$ we have $d_X(x, y) = d_X(y, x)$ (**Symmetry**)
(3) $\forall x, y, z \in X$ we have $d_X(x, y) \leq d_X(x, z) + d_X(z, y)$ (**Triangle inequality**)

Problem 2. Decide whether or not the following maps are valid metrics. Sketch, whenever possible, the corresponding unit circle.

(1) $X = \mathbb{R}^2$ and let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $d_X(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2$
(2) $X = \mathbb{R}^2$ and let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $d_X(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
(3) $X = \mathbb{R}$ and $d_X(x, y) = |x^2 - y^2|$
(4) $X = \mathbb{R}^2$ and let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $d_X(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

Definition 3. Let X be a metric space with metric d_X. Then:

(1) We define a **neighborhood** of radius ε and center $x \in X$ as the set $\mathcal{B}_\varepsilon(x) = \{y \in X \text{ such that } d_X(x, y) < \varepsilon\}$
(2) $p \in X$ is a **limit point** for X if and only if for any $\varepsilon > 0$ there exist $y \neq p$, such that $y \in \mathcal{B}_\varepsilon(p) \cap X$
(3) Given a subset E of a metric space, that is $E \subset X$, we say that $p \in E$ is an **interior point** of E if and only if there exists some neighborhood of p entirely contained in E
(4) A subset E of a metric space X, that is $E \subset X$, is said to be **open** if and only if any point of E is an interior point of E
(5) A subset E of a metric space X, that is $E \subset X$, is said to be **closed** if and only if it contains all of its limit points

Problem 4. Show that any neighborhood is an open set

Problem 5. Show that if a metric space X has a limit point p, then X must have infinitely many elements

Problem 6. Give an example of a set with one limit point which does not belong to the set itself. Is this set closed?

Problem 7. Give an example of a set with exactly one limit point.

Problem 8. Give an example of a set with infinitely many points, but with no limit points