The Ordinal Numbers

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1 Well orderings

An order on a set $<$ satisfies that if $x < y$ then $y \not< x$ and if $x < y$ and $y < z$ then $x < z$. A total order is one that either $x = y$, $x < y$ or $y < x$ for every pair of elements x and y. A well order is a total order such that every nonempty subset has a least element. An example of a well order is $\mathbb{N}$ (all positive whole numbers), non-examples are $\mathbb{Z}$ and $\mathbb{Q}^+$ (these are the whole numbers, positive and negative, and the positive rational numbers, respectively. The set of all rational numbers isn't well-ordered either, but I chose the positive ones to demonstrate the difference between it and the whole numbers).

1.1 The initial segment

Given a well-ordering on a set W we define, for any $x \in W$ the initial segment $W(x) := \{w \in W : w < x\}$. We also define a function $f : W_1 \to W_2$ from one ordered set to another to be order-preserving if $x < y \implies f(x) < f(y)$. An *order isomorphism* is a bijection (1-to-1 function, that takes every element to exactly one element) that is order preserving, we denote the existence of an ordered isomorphism between two ordered sets W_1, W_2 as $W_1 \cong W_2$.

Theorem 1. *For any well ordered set W, and any $x \in W$, $W(x) \ncong W$*

Proof. Assume by way of contradiction that $W(x) \cong W$. Let $f : W \to W(x)$ be an order preserving function, then we automatically have $f(x) \in W(x)$. But by definition of $W(x)$ this gives us $x > f(x)$. Now define $S = \{w \in W : f(w) < w\}$. Since $x \in S$, we have S nonempty, so it has a least element, call this y. Now, we have $f(y) < y$, and that f is order preserving, so $f(f(y)) < f(y)$, giving us $f(y) \in S$. But we had $f(y) < y$, so that this contradicts the minimality of y. ■

(you may notice we didn't need to assume that f was a bijection in this proof)

1.2 Giving order to order

We are now ready to prove an important result.

Theorem 2. *Given two well-ordered sets W_1, W_2 exactly one of the following holds:* (i) $W_1 \cong W_2$ (ii) $W_1(x) \cong W_2$ *(for some $x \in W_1$), or* (iii) $W_1 \cong W_2(y)$ *(for some $y \in W_2$)*

Proof. Clearly, if we had (i) and (ii), this would give $W_1 \cong W_1(x)$, a contradiction, and similarly for (i) and (iii). Now assume we had (ii) and (iii). Then we would have order preserving bijections $f : W_1 \to W_2(y)$ and $g : W_2 \to W_1(x)$. Consider the composite map $g \circ f : W_1 \to g[W_2(y)]$, this is obviously an order-preserving bijection, since it's a composition of two of these. We now show that $g[W_2(y)] = W_1(z)$ for some $z \in W_1$, which will give us the desired contradiction. We have $\forall w \in g[W_2(y)] : w \in W_1(x) \implies w < x$ so that $S = \{s \in W_1 : \forall w \in g[W_2(y)] : s > w\}$ is nonempty, then let z be the least element of S, we show that $g[W_2(y)] = W_1(z)$. Since by definition of $z \in S$ we have $g[W_2(y)] \subset W_1(z)$, we need only show that $g[W_2(y)] \supset W_1(z)$ assume to the contrary, that some $w \in W_1(z)$ is not mapped to, then take the least such w. We have some $w' > w$ that is mapped to, for otherwise w would be the least element of S. But since $z \leq x$, we have $W_1(z) \subset W_1(x)$ so that w is mapped to by some element $a \in W_2$, but since w is not in the image of $W_2(y)$, we have $a \geq y > g^{-1}(w')$, but then we'd need to have $w = g(a) > w'$, by the order preserving property of g. But we chose w' such that $w' > w$, a contradiction. So we have $g[W_2(y)] = W_1(z)$ for some z which gives $W_1 \cong W_1(z)$, a contradiction.

Now we show that at least one of (i), (ii) or (iii) holds. Define $f = \{(x, y) \in W_1 \times W_2 : W_1(x) \cong W_2(y)\}$. Then we cannot have $(x_1, y), (x_2, y) \in f$ for $x_1 \neq x_2$ because then we'd have $W_1(x_1) \cong W_1(x_2)$ (impossible because one of $x_1 > x_2$ or $x_2 > x_1$ must hold, and thus one of these is contained in the other, giving an order isomorphism between something and its initial segment), similarly for any $y_1, y_2 \in W_2$. This gives us that if f were a function (i.e. if it had everything for W_1 as a first coordinate, or everything from W_2 as a second coordinate) then it would be injective. We show that this is the case. Say that there was $x \in W_1$ **and** $y \in W_2$ that didn't appear in f. Then choose least such x and y, but we would have $W_1(x) \cong W_2(y)$ under f (since we already showed that f is well defined, and it is trivially order-preserving), a contradiction. Thus at least one of (i), (ii) or (iii) holds, with an order isomorphism given by f. ∎

2 The Ordinals

We will now introduce the collection of all well-ordered sets, but first some preliminary notions:

2.1 Transitive Sets

A *transitive set* is a set S such that $\forall x \in S : x \subset S$. For example, the rational numbers $\mathbb{Q}$, when viewed as Dedekind cuts, are a transitive set (note that not all subsets are elements, which must be the case by Cantor's theorem). The reason this is called transitive is because it turns element containment into a transitive relation i.e. if z is a transitive set, then $x \in y \in z \implies x \in z$.

2.2 The Ordinals

We are now ready to introduce our main topic: the ordinals. A set α is an ordinal if and only if α is transitive and well ordered by the relation given by $\in$. We mean by this that is well ordered by the ordering $x < y \Leftrightarrow x \in y$. The ordinals are essentially the primitive or canonical well-ordered sets, as their ordering comes only from set containment.

Theorem 3. *If α is an ordinal and $\beta \in \alpha$ then β is an ordinal as well.*

Proof. We have $\beta \subset \alpha$, so that it inherits its well ordering from α so we need only show that β is transitive. Let $\gamma \in \beta$ then for any $\delta \in \gamma$ we have $\delta < \gamma$, but orderings are transitive so that $\delta < \beta$. This gives us $\delta \in \beta$ so that $\gamma \subset \beta$, as desired. ∎

2.3 Ordering the Ordinals

Theorem 4. *The intersection of two ordinals is an ordinal*

Proof. The intersection is well ordered, as a subset of the well ordered set α. Now if $x \in \alpha \cap \beta$ then $x \in \alpha$ and $x \in \beta$ so that $x \subset \alpha$ and $x \subset \beta$ but this gives $x \subset \alpha \cap \beta$. Thus the intersection is transitive and well-ordered, thus an ordinal. ∎

Theorem 5. *For any two ordinals α,β. We have exactly one of $\alpha = \beta$, $\alpha < \beta$ or $\alpha > \beta$.*

We omit the proof because it is complicated and not that enlightening.

Theorem 6. *A subset of the collection of ordinals (not necessarily an ordinal itself), has a least element.*

Proof. Take any $\alpha \in A$ then consider $\alpha \cap A$, if this intersection is empty, then nothing less than α is in A and so α is least in A. If it is nonempty, then it is a subset of α and so has a least element by the fact that α is well ordered. ∎

2.4 What are the Ordinals

Call Ord the collection of ordinals.

We have trivially that $\emptyset \in Ord$. Now, given an element $x \in Ord$ we also have $x \cup \{x\} \in Ord$ Since this remains transitive, and we only change it's ordering by adding a new greatest element, namely x. Given a collection A of ordinals, we can define an ordinal $\sup A := \alpha : \forall a \in A, \alpha > a$. This gives us the limit ordinals, which are given only as a limit and not as the successor of the thing before it. The first limit ordinal is $\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots\}$

We can then consider $\omega + 1 = \omega \cup \{\omega\}$ Etc.

3 Ordinal arithmetic

We are now ready to consider arithmetic on the ordinals. Which give us ways of obtaining new ordinals.

3.1 Ordinals as Numbers

We view the finite ordinals as the regular natural numbers, namely: $0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}, \ldots$. We define $x + 1$ as $x \cup \{x\}$.

3.2 Addition

We define addition recursively by:

- $\alpha + 0 = \alpha$
- $(\alpha + \beta) + 1 = \alpha + (\beta + 1)$
- $\alpha + \beta = \sup\{\alpha + \gamma : \gamma < \beta\}$ for limit ordinals β

Consider for example $n + \omega$ this gives us $\sup\{n + m : m \in \omega\}$ but this is just the ordinal greater then all sums of things in ω, while every sum of things in ω is again in ω, so this is just ω itself. However, when we consider $\omega + n$ this is equal to $\sup\{\omega + m : m < n\}$ which is the set $\underbrace{\{\omega, \omega \cup \{\omega\}, \omega \cup \{\omega, \{\omega\}\}, \ldots\}}_{n\text{-times}}$. In particular, this tells us that ordinal arithmetic isn't commutative.

3.3 Multiplication

We define multiplication as:

- $\alpha * 0 = 0$
- $\alpha * (\beta + 1) = \alpha * \beta + \alpha$
- $\alpha * \beta = \sup\{\alpha * \gamma : \gamma < \beta\}$ for limit ordinals β

This gives us $\alpha * 1 = \alpha$ and $\alpha * 2 = \alpha + \alpha$ and so $\alpha * n = \underbrace{\alpha + \alpha + \alpha + \ldots}_{n\text{-times}}$

Theorem 7. $1 * \alpha = \alpha$

Proof. We proceed by transfinite induction:

- $1 * 0 = 0$
- $1 * (\alpha + 1) = \alpha + 1$
- $1 * \alpha = \sup\{1 * \beta : \beta < \alpha\} = \sup\{\beta : \beta < \alpha\} = \alpha$ for limit ordinals α

∎

Again, multiplication isn't commutative, as $2 * \omega = \omega$ while $\omega * 2 = \omega + \omega = \sup\{\omega + \alpha : \alpha < \omega\}$

3.4 Exponentiation

We define exponentiation as:

- $\alpha^0 = 1$
- $\alpha^{\beta+1} = \alpha^\beta * \alpha$
- $\alpha^\beta = \sup\{\alpha^\gamma : \gamma < \beta\}$

We have $1^\omega = \sup\{1^n : n \in \omega\} = \sup 1 = 1$ which is unsurprising, and holds for all ordinals. However, consider $2^\omega = \sup\{2^n : n \in \omega\} = \sup\{n : n \in \omega\} = \omega$, this is in contrast to the cardinal numbers, where for every cardinal $\aleph$ we have $2^{\aleph_n} \geq \aleph_{n+1}$. The same holds for every n, namely $n^\omega = \omega$, however, this is not true for $\omega^\omega = \sup\{\omega^n : n \in \omega\}$, because consider $\omega^2 = \omega * \omega = \sup\{\omega * n : n \in \omega\}$ so that $\forall n \in \omega : \omega^2 > \omega * n$.

Given that $\omega^\omega > \omega$ we also have that $\omega^{(\omega^\omega)} > \omega^\omega$ and so we can consider the sequence $\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$, specifically, its supremum, which we call ε. We thus have the equation $\omega^\epsilon = \epsilon$.

3.5 The Ordinals

We can now picture the ordinals all together, they look something like:

$$
\begin{aligned}
&0, 1, 2, \ldots \omega, \omega + 1\, \omega + 2, \ldots \omega * 2, \omega * 2 + 1, \ldots \omega * 3, \ldots \omega^2, \omega^2 + 1 \ldots \omega^2 + \omega, \\
&\omega^2 + \omega + 1, \ldots \omega^3, \ldots \omega^3 + \omega, \ldots \omega^3 + \omega * 2, \ldots \omega^3 + \omega^2, \omega^3 + \omega^2 + 1, \ldots \\
&\omega^\omega, \ldots \omega^\omega + \omega, \ldots \omega^\omega * 2, \omega^\omega * 2 + 1, \ldots \omega^{\omega+1}, \ldots \omega^{\omega^2}, \ldots \omega^{\omega^2+1}, \\
&\ldots \omega^{\omega^2+\omega}, \ldots \omega^{\omega^3}, \ldots \omega^{\omega^\omega}, \ldots \varepsilon, \varepsilon + 1 \ldots
\end{aligned}
$$