11483: Introduction to Modern Physics
Lecture-2

Chitraang Murdia

July 13, 2017

1 Special Relativity : Applications

1.1 Events and Spacetime Interval

An event in relativity is characterized by a location and a time. Two simultaneous events occur at the same time in a given frame. Two collocated events occur at the same location in a given frame.

In classical mechanics, distances and time intervals are frame independent i.e. an object's length or a time interval does change as we go from one inertial frame to another. However, this is not true in relativistic mechanics as the Lorentz transformations mix space and time.

The invariant quantity in relativity is spacetime interval

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{1}$$

This quantity does not change as we transform from one inertial frame to another.

$$\begin{aligned}
\Delta s'^2 &= -c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2 &(2)\\
&= -c^2\left(\frac{\Delta t - u/c^2\Delta x}{\sqrt{1-u^2/c^2}}\right)^2 + \left(\frac{\Delta x - u/\Delta t}{\sqrt{1-u^2/c^2}}\right)^2 + \Delta y^2 + \Delta z^2 &(3)\\
&= \frac{(-c^2\Delta t^2 + 2u\Delta x\Delta t - u^2/c^2\Delta x^2) + (\Delta x^2 - 2u/\Delta t\Delta x + u^2\Delta t^2)}{1-u^2/c^2} &(4)\\
&\quad + \Delta y^2 + \Delta z^2 &(5)\\
&= -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 &(6)\\
&= \Delta s^2 &(7)
\end{aligned}$$

Two events are time-like separated if $\Delta s^2 < 0$, space-like separated if $\Delta s^2 > 0$, and light-like or null separated if $\Delta s^2 = 0$.

Two events on the path of a massive object are time-like separated because

$$\frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{\Delta t^2} = v^2 < c^2 \tag{8}$$

$$-c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 < 0 \tag{9}$$

Two events on the path of a photon are light-like separated because

$$\frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{\Delta t^2} = c^2 \tag{10}$$

$$-c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = 0 \tag{11}$$

Since Lorentz transformations can change the time interval between two events, a natural question arises: Does relativity violate causality? Causality is the central idea that cause must always precede effect. However for two events to be cause and effect, it must be possible for some signal to travel between them, so

$$\frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{\Delta t^2} \leq c^2 \tag{12}$$

$$-c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \leq 0 \tag{13}$$

Hence, the events are time-like or light-like separated. If the events are not collocated,

$$\Delta x^2 + \Delta y^2 + \Delta z^2 > 0 \tag{14}$$

$$\Delta t^2 > 0 \tag{15}$$

Hence, Δt cannot change sign because it never becomes zero (see Appendix A). If cause precedes effect in one inertial reference frame, it must always precede effect in any other inertial reference frame. Thus, causality is not violated.

Note that if objects could travel at velocities higher than c, this would violate relativity because then points on its path would be space-like separated and so it would be possible to flip the order of cause and effect. Hence, causality is the reason why objects cannot travel faster than light.

1.2 Length Contraction

The length of an object is measured by measuring the location of its endpoints simultaneously. The length of an object in its rest frame is called the proper length.

Consider a rod at rest in frame F. The endpoints of this rod are located at $x_1 = 0$ and $x_2 = L$, so its proper length is $\Delta x = x_2 - x_1 = L$. Now consider the frame F' that moves at a speed u with respect to F as above.

If we want to measure the length of the rod in F', we need to measure the difference between its endpoints, $\Delta x'$ simultaneously i.e. $\Delta t' = 0$.

From the Lorentz transformations we have

$$\Delta x = \frac{\Delta x'}{\sqrt{1 - u^2/c^2}} \tag{16}$$

$$\Delta x' = \Delta x \times \sqrt{1 - u^2/c^2} = L/\gamma(u) < L \tag{17}$$

Thus, the length of the object decreases in other frames. This phenomenon is called length contraction.

1.3 The Barn-Ladder Paradox

Consider a barn of proper length L_b and a ladder of proper length $L_l > L_b$. Now let us move the ladder at a very high velocity through the barn.

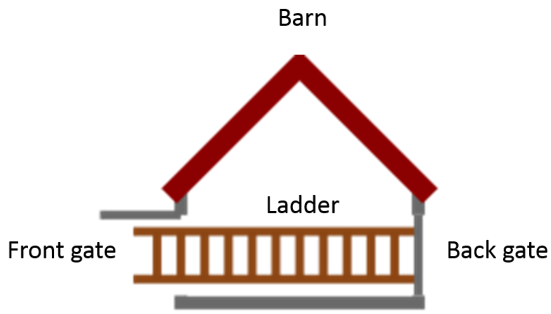

Figure 1: Schematic Diagram of the Setup

From the barn's frame of reference, the ladder is length contracted and it is possible to fit the ladder in the barn. Let us imagine two guards sitting at the two ends of the barn. The one at the front end closes the front gate the moment back end of the ladder enters that gate. The one at the back end opens the gate the moment the front end of the ladder reaches that gate. Thus, in the barn's frame there is a point in time in which the ladder is contained within the barn with both the gates closed.

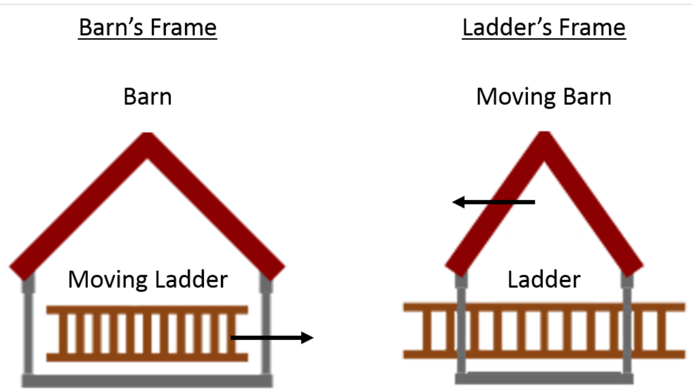

Figure 2: Schematic Diagram of the Paradox

From the ladder's reference frame, the barn is length contracted and there is no way that the ladder can fit in the barn. This is the paradox.

The resolution comes from the fact that the closing of the front gate and the opening of thee back gate are space-like separated. They occur in one order in the barn's frame and in another order in the ladder's frame. Note that these events are not a cause and effect pair.

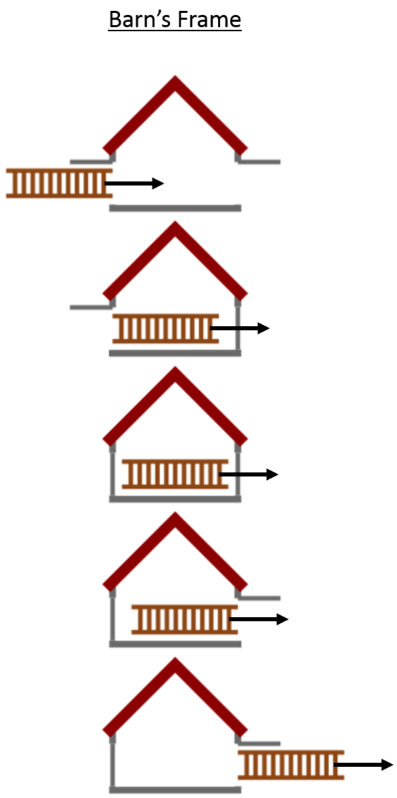

Figure 3: Schematic Diagram of the Resolution

A more involved version of the paradox deals with the case where the back gate is never opened. In the barn's frame the back end of the ladder enters the front gate before the front end of the ladder hits the back gate, so the ladder can be trapped within the barn by closing the front gate. This is clearly a paradox.

The paradox in here arises because we assume that the ladder is a rigid body so all parts of the ladder must move with the same velocity. However, this requires that an acceleration/deceleration information needs to be transmitted from one end of the ladder to the other end simultaneously. This is clearly impossible because information cannot travel at a speed more than c. Hence, the rigidity is not consistent with relativity i.e. there are no rigid objects in actuality.

1.4 Time Dilation

Consider two collocated events in frame F $x_1 = 0,\ t_1 = 0$ and $x_2 = 0,\ t_2 = \tau$. The time interval between these events is $\Delta t = \tau$. This is called the proper time.

The time interval between these events in F' above is

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - u^2/c^2}} = \gamma(u)\tau > \tau \tag{18}$$

Thus, the length of the time interval increases in other frames. This phenomenon is called time dilation.

1.5 The Twin Paradox

Consider that we have a pair of twins, P and Q. P decides to stay at rest (on a space station) while Q travels to a far-away galaxy at a very high but constant velocity.

From P's perspective Q is moving so time dilation occurs for Q and P ages more than Q. Note that time dilation implies that Q's heartbeats are more spread out as compared to P so he ages less in any given duration. The reverse happens from Q's perspective.

Q now reverses direction and moves towards P. As before P ages more from his perspective and Q ages more from Q perspective. The paradox arises when p and Q meet and we need to decide who is more old.

The resolution comes from the fact that Q's frame is not an inertial frame of reference because he accelerated at some point in his journey. Q's perspective is not entirely correct because time dilation does not apply when he is accelerating. As P's perspective is correct, P is older.

1.6 Doppler Shift

The waves emitted by a stationary source have the same frequency in all directions because the wavefronts form concentric spheres. However for a moving source the wavefronts squish in along the direction of motion and spread out opposite to the direction of motion. This phenomenon is called Doppler Effect.

The squishing in leads to an increase in frequency and this phenomenon is called blueshift. The squishing in leads to a decrease in frequency and this phenomenon is called redshift.

Consider a source of frequency f_0 moving towards a detector with velocity v. Needless to say v is negative if the source is moving away from detector. The frequency detected by the detector is given by the relativistic Doppler shift formula to be

$$f = f_0 \sqrt{\frac{1 + v/c}{1 - v/c}} \tag{19}$$

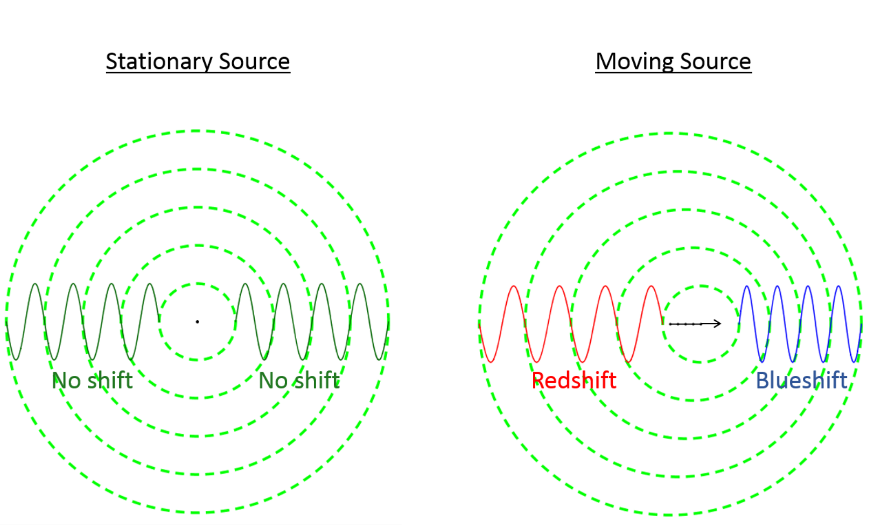

Figure 4: Doppler Shift

A Bolzano's Theorem

If a continuous function has values of opposite sign inside an interval, then it has a root in that interval. This is because there is no way a continuous function can change signs without intersecting the x-axis.

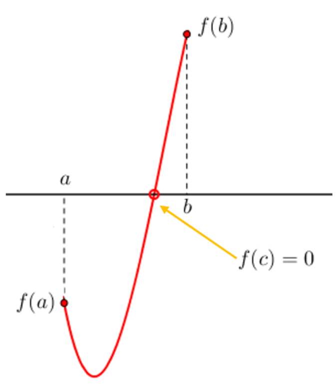

Figure 5: Representative sketch for Bolzano's Theorem

B Waves: Terminology and Classification

A wave is an oscillation accompanied by a transfer of energy and momentum. Some important terms related to waves are as below:

- Wavefunction (ψ): The wavefunction is the physical property that oscillates due to the wave.
- Period (T): The period is the time it takes a wave to complete one oscillation.
- Frequency (f): The frequency is the number of oscillations that are completed per unit time.

$$f = 1/T$$

- Wavelength (λ): The wavelength is the distance over which a wave repeats itself. Alternatively, it is the distance between two successive crests or two successive troughs.

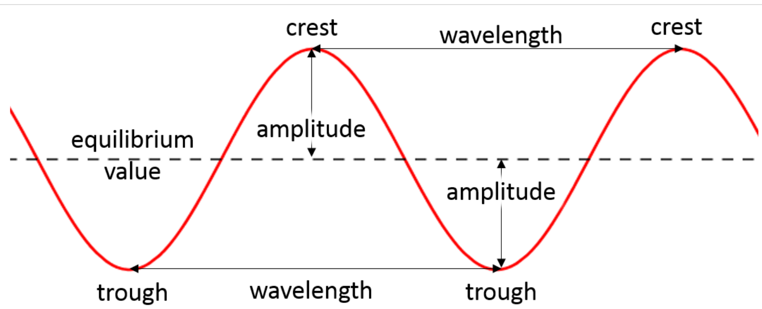

Figure 6: Sketch of a wave

- Wavevector (k): This is simply 2π divided by the wavelength.

$$k = 2\pi/\lambda$$

- Angular frequency (ω): This is simply 2π time the frequency.

$$\omega = 2\pi f = 2\pi/T$$

- Wavefront: The wavefront is the set of points affected in the same way by a wave at a given instant.
- Speed (c_0): This is the speed at which the wavefronts propagate.

$$c_0 = \lambda/T = \lambda f = \omega/k$$

- Amplitude (A): This is the maximum displacement of the wavefunction from its equilibrium value.
- Intensity (I): This is the energy carried by the wave per unit area. The intensity is proportional to amplitude squared.

$$I \propto A^2$$

Wave can broadly be classified into two types:

1. Mechanical wave: A mechanical wave is a wave that is an oscillation of matter. It requires a material medium for propagation.
2. Electromagnetic wave: An electromagnetic wave is a wave of the electromagnetic field, The wavefunction here is the electric or magnetic field. It does require a material medium for propagation. All electromagnetic waves move at speed c through a vacuum.

C Waves: Electromagnetic Spectrum and Photons

Electromagnetic waves are classified into the electromagnetic spectrum based on the frequency/wavelength:

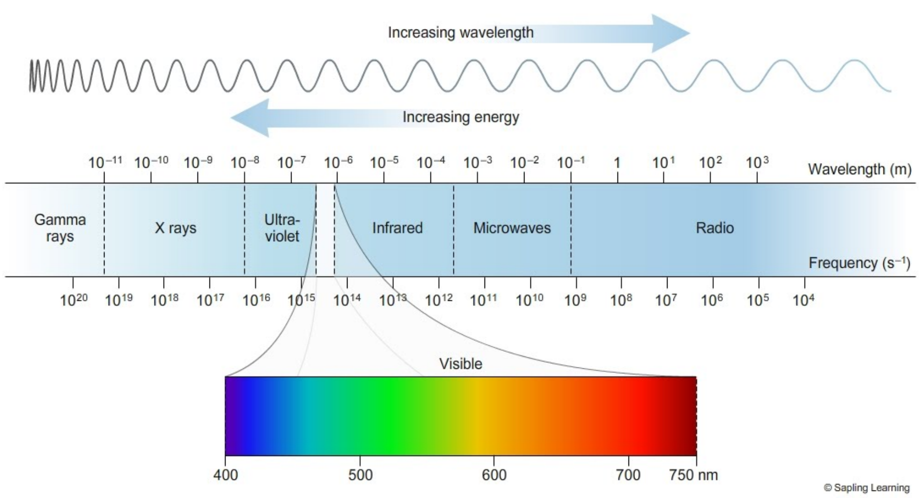

Figure 7: Electromagnetic spectrum

In quantum mechanics, electromagnetic waves are composed of packets or quanta called photons. These photons are massless particles that travel at speed c in vacuum.

The energy of a photon is

$$E = hf = \hbar\omega \tag{20}$$

where $h = 6.62607 \times 10^{-34} Js$ is Plank's constant and $\hbar = h/2\pi = 1.05457 \times 10^{-34} Js$ is reduced Plank's constant.

The momentum of a photon is

$$p = h/\lambda = \hbar k = \hbar\omega/c \tag{21}$$

The energy-momentum relation (dispersion relation) for a photon is

$$E = pc \tag{22}$$