Building with spheres

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November 19, 2017

1 Algebraic topology

A fundamental question in topology is when two shapes are similar, or perhaps to what extent two shapes are similar. A very useful tool for doing this type of analysis is decomposing shapes into other shapes. We will be doing this type of decomposition, and hopefully in the process relay some of the intuition involved in showing differences between shapes. The primary building blocks are spheres and balls, a sphere is the surface of a ball. We have one ball and 1 sphere for each dimension.

One thing to keep in mind is that we are allowed to deform shapes, so that square=triangle=circle=blob. We are focusing on more fundamental properties of shapes, so a simple deformation isn't enough for us to consider two shapes different. However, triangle≠dot, because a triangle has a hole while a dot doesn't.

2 Balls and spheres

A way to concretely describe a sphere is the set of points a set distance from a center point in a certain dimension. A ball is similar, except instead we take the points less than or equal to a certain distance. The point here is that this is somewhat easy to visualize for the first few dimensions, and we don't concern ourselves with the fact that it's harder to visualize in higher dimensions, we just rely on our intuition from the cases that we know.

One way to construct a sphere in dimension n is to take the ball in dimension $n-1$ and collapse the boundary (i.e. the sphere in dimension $n-1$) to a point. For example, the "ball" in one dimension is just a line, and its boundary is the two endpoints. If we take these two points and combine them into a single point, we get a circle. Similarly, if we take a disk, and collapse it's border, the circle, into a point, we get a sphere. We will get back to the 4d sphere a little later.

3 Cell complexes

A cell complex is a thing built out of various balls of different dimensions. We start with a bunch of points, and add 1d balls, which are just lines, in a way that the sphere on the edge, which is just two points, gets put into those original points. Then we add disks, in so that the boundary sphere, which is a circle, gets put into the thing so far built.

One example is a hollow tube, we begin with two points, attach two circles and a line between the points, and then attach a disk to make a tube.

In general we can make a sphere out of the previous sphere by attaching two balls. For a circle we attach two disks, one becomes the upper hemisphere and the other the lower. In this way we can imagine adding two balls to a sphere, one on the inside, and one turned insides out on the outside. This leads us to adding a "point at infinity" attaching the two halves of that ball. This is not actually what a 4d sphere looks like, but it's a way of imagining it.

4 Wedge sum

We can take two shapes and attach them at a point, we call this the wedge sum of the two. For example the wedge sum of two circles is a figure 8.

5 Quotient

If we take a shape, and some shape inside that shape (a sub-shape) we can take the quotient by collapsing the shape into a point. For example, taking the quotient of a sphere by a circle gives us the wedge sum of two spheres. We can in general do this with any sphere and a sphere of one dimension lower. In order to understand this, remember our cell complex picture of a sphere as two balls attached to a sphere of a dimension lower. If we collapse that sphere, it's the same as collapsing the sphere border of the two balls, which gives two spheres, attached at a point, where the border used to be.

Another similar type of quotient is any perhaps more complicated rule for combining points. For example, if we take the circle and attach opposite points, we get a circle again.

6 The torus

We now come to a more interesting shape, the torus, which is the surface of a donut. One way of creating a torus is by taking a cylinder and attaching the two opposite ends together. Using this, we can construct the torus as a cell complex, following a similar procedure to the cylinder, but by attaching the two opposite ends at the start. This amounts to attaching a disk to the wedge sum of two circles. This is also the product of two circles, since we get a circles for every point of the circle.

We can also view the cylinder as a product of a line and a circle. And the ideal is that we can first collapse the second interval into a circle and then take the product, or first take the product and then collapse the interval. Similarly, we can start with a square - and interval times itself, and then collapse the two intervals into circles to get the torus.

7 Solid cylinder

A solid cylinder is a product of an interval with a disk. It is also a cell complex of its boundary filled in with a solid sphere. The cell complex starts with two points, we then attach a circle to each and a line between them. We then attach three disks to get the edge of a cylinder. then map the edge of a sphere to the cylinder and fill it in with the interior of the sphere.

Notice that in the product of the interval with a disk, if we take the boundary of the interval times the disk we get two disks, which is part of the boundary of the solid cylinder. Similarly the boundary of the disk is the circle, and multiplying this by the interval gives a tube, these together are the boundary of the solid cylinder, a hollow cylinder.