INTRODUCTION TO LEBESGUE MEASURE: HANDOUT

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This handout has a quick guide to notation I will be using in class without defining and two fairly technical examples that I may not have time to do fully in class.

1. Notation

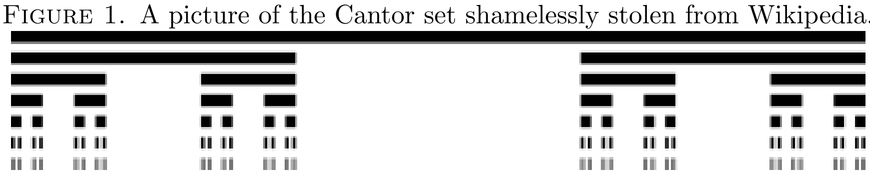
Some notation I will be using throughout the class:

- $\{x : P(x)\}$ is the set of points x that have some property $P(x)$. More generally, $\{f(x) : P(x)\}$ means the set of expressions $f(x)$ where x is such that $P(x)$ holds. For example, $\{x : x > 2\}$ is the set of numbers that are greater than 2 an $\{x^2 : x > 2\}$ is the set of squares of numbers that are greater than 2 (which is just equal to the set of numbers that are greater than 4).
- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{Q}$ is the set of rational numbers.
- If A and B are sets, then their *difference* (or the *complement* of B in A) is
$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$
- If $A_1, A_2, A_3, \ldots$ are sets, then we write
$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \ldots$$
for their infinite union.
- For $a < b$ real numbers,
$$\begin{aligned} [a, b] &= \{x : a \le x \le b\} \\ [a, b) &= \{x : a \le x < b\} \\ (a, b] &= \{x : a < x \le b\} \\ (a, b) &= \{x : a < x < b\} \end{aligned}$$
Sets of this form are called *intervals* and their *length* is $b - a$.

2. The Cantor set

The Cantor set is an example of a fairly complicated Borel set whose measure can be computed. Intuitively, we define the Cantor set as follows. Start with the interval $[0, 1]$, and then remove the middle third of the interval to get two intervals $[0, 1/3] \cup [2/3, 1]$. Then remove the middle thirds of each of those two intervals to get four intervals, and then remove the middle thirds of each of the four intervals to get eight intervals, and so on. What you're left with after doing this infinitely many times is called the Cantor set K. See Figure 1 for what the first few steps of this look like.

FIGURE 1. A picture of the Cantor set shamelessly stolen from Wikipedia.



More precisely, we define a sequence of finite unions of intervals A_n such that A_n is the union of the middle thirds we remove at the nth stage of this process. In particular, we have:

$$\begin{aligned} A_1 &= (1/3, 2/3) \\ A_2 &= (1/9, 2/9) \cup (7/9, 8/9) \\ A_3 &= (1/27, 2/27) \cup (7/27, 8/27) \cup (19/27, 20/27) \cup (25/27, 26/27) \\ &\dots \end{aligned}$$

In general, A_n is a union of 2^{n-1} intervals, each of which has length $1/3^n$. If we really wanted to, we could write down an explicit formula for what A_n is, but it would be messy and complicated so I won't.

We now define $A = \bigcup A_n$ and define the Cantor set to be the complement $K = [0, 1] - A$. Since each A_n is a finite union of intervals, A is a Borel set, so K is a Borel set. Furthermore, the sets A_n are disjoint and $\lambda(A_n) = 2^{n-1}/3^n$, so

$$\lambda(A) = \sum \lambda(A_n) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}.$$

If you've seen the formula for geometric series, you can compute that this sum is

$$\lambda(A) = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1.$$

Thus $\lambda(K) = \lambda([0, 1]) - \lambda(A) = 1 - 1 = 0$. This sort of makes sense, because K is a sort of infinitely scattered "dust" that we wouldn't expect to have any length. However, we could have chosen the lengths of the intervals we removed to be smaller so that the measure was positive, even though the set would still look essentially the same!

Also, the Cantor set is an example of a set which is uncountable but still has measure 0. You can show that K is uncountable by observing that K is exactly the set of numbers that have a base 3 expansion containing only the digits 0 and 2. This is because A_1 is the set of numbers whose first base 3 digit is 1, A_2 is the set of numbers whose second base 3 digit is 1, and so on. Thus for every infinite sequence of 0s and 2s, we get an element of the Cantor set by considering the number having that base 3 expansion. By a diagonalization argument similar to the proof that the real numbers are uncountable, the set of such infinite sequences is uncountable.

For more information about the Cantor set and its many fascinating properties, a good starting point is the Wikipedia page `http://en.wikipedia.org/wiki/Cantor_set`.

3. AN UNMEASURABLE SET

Here is a construction of a set on which Lebesgue measure cannot be defined. First, recall the *translation-invariance* property of Lebesgue measure:

Theorem. *If $A \subseteq \mathbb{R}$ is measurable and $x \in \mathbb{R}$, then $A + x = \{a + x : a \in A\}$ is measurable and $\lambda(A + x) = \lambda(A)$.*

Intuitively, $A+x$ is just A "moved to the right" by x on the line, so the measure shouldn't change (in two dimensions, this is related to the fact that congruent sets have the same area).

Now we will construct a set whose measure cannot be defined in any way that is consistent with translation-invariance. Let $A = [0, 1)$ and for each $x \in A$, define

$$Q_x = \{y \in A : x - y \in \mathbb{Q}\}.$$

Now I claim that for any $x, y \in A$, either $Q_x \cap Q_y = \emptyset$ or $Q_x = Q_y$. Indeed, suppose $Q_x \cap Q_y$ is nonempty and let z be some element of it. Then $x - z$ and $y - z$ are rational, so $y - x = (y - z) - (x - z)$ is rational. Let $w \in Q_x$, so $x - w$ is rational. Then $y - w = (y - x) + (x - w)$ is rational, so $w \in Q_y$. By a similar argument we can show that if $w \in Q_y$ then $w \in Q_x$, so we conclude that $Q_x = Q_y$.

Now from each set Q_x pick a single element and form a new set B from these elements. That is, B is a set such that $B \cap Q_x$ contains exactly one point for each x. This is possible because distinct sets Q_x are disjoint.

Now for each $q \in \mathbb{Q} \cap A$ we define a subset $B_q \subseteq A$ as follows. First consider $B + q \subseteq A + q = [q, q + 1)$. We want to just say $B_q = B + q$, but we can't because $B + q$ may not be contained in $[0, 1)$. So what we do is translate back the part of $B + q$ that is bigger than 1. More precisely, let $C_q = (B + q) \cap [q, 1)$ and $D_q = (B + q) \cap [1, 1 + q)$ and define

$$B_q = C_q \cup (D_q - 1).$$

Note that $C_q \subseteq [q, 1)$ and $D_q - 1 \subseteq [0, q)$ so they are disjoint. Now suppose we could define the measure of B. Then by translation-invariance, for each q we would have:

$$\begin{aligned} \lambda(B_q) &= \lambda(C_q) + \lambda(D_q - 1) \\ &= \lambda(C_q) + \lambda(D_q) \\ &= \lambda(B + q) \\ &= \lambda(B). \end{aligned}$$

But now I claim that every point $x \in A$ is in B_q for exactly one value of q. Indeed, B contains exactly one point y from Q_x, and $x - y \in \mathbb{Q}$. If $x \geq y$, then we let $q = x - y$ and then $x = y + q \in D_q \subseteq B_q$, and if $x < y$, we let $q = x - y + 1$ and then $x = y + q - 1 \in C_q - 1 \subseteq B_q$. Conversely, if $x \in B_q$, then $x - q$ or $x - q + 1$ must be in B and hence must be y (since $x - q$ or $x - q + 1$ is in Q_x), so this is the only q that works.

Thus we can split up $A = \bigcup_{q \in \mathbb{Q} \cap A} B_q$ as a disjoint union. By countable additivity (since there are only countably many q),

$$\lambda(A) = \sum_q \lambda(B_q).$$

Now $\lambda(A) = 1$ since A is just $[0, 1)$, and $\lambda(B_q) = \lambda(B)$ for all q, so this is just

$$1 = \sum_q \lambda(B) = \lambda(B) + \lambda(B) + \lambda(B) + \dots.$$

If $\lambda(B) = 0$, the right-hand side is 0, and if $\lambda(B) > 0$, the right-hand side is infinite. Thus there is no possible value of $\lambda(B)$ that makes this equation hold! From this we conclude that B is *unmeasurable*–its measure cannot be defined.

Intuitively, what's going on is that we've split $[0, 1)$ into countably many sets, all of which must have the same size because they're all just the same set broken into pieces and translated. This is impossible by countable additivity, because there is no number that you can add together infinitely many copies of to get 1.