

O. Linear Differential Operators

1. Linear differential equations. The general linear ODE of order n is

$$(1) \quad y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y = q(x).$$

If $q(x) \neq 0$, the equation is **inhomogeneous**. We then call

$$(2) \quad y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y = 0.$$

the **associated homogeneous** equation or the *reduced* equation.

The theory of the n -th order linear ODE runs parallel to that of the second order equation. In particular, the general solution to the associated homogeneous equation (2) is called the **complementary** function or solution, and it has the form

$$(3) \quad y_c = c_1y_1 + \dots + c_ny_n, \quad c_i \text{ constants,}$$

where the y_i are n solutions to (2) which are *linearly independent*, meaning that none of them can be expressed as a linear combination of the others, i.e., by a relation of the form (the left side could also be any of the other y_i):

$$y_n = a_1y_1 + \dots + a_{n-1}y_{n-1}, \quad a_i \text{ constants.}$$

Once the associated homogeneous equation (2) has been solved by finding n independent solutions, the solution to the original ODE (1) can be expressed as

$$(4) \quad y = y_p + y_c,$$

where y_p is a particular solution to (1), and y_c is as in (3).

2. Linear differential operators with constant coefficients

From now on we will consider only the case where (1) has constant coefficients. This type of ODE can be written as

$$(5) \quad y^{(n)} + a_1y^{(n-1)} + \dots + a_ny = q(x);$$

using the differentiation operator D , we can write (5) in the form

$$(6) \quad (D^n + a_1D^{n-1} + \dots + a_n)y = q(x)$$

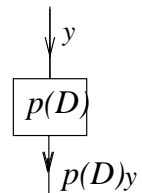
or more simply,

$$p(D)y = q(x),$$

where

$$(7) \quad p(D) = D^n + a_1D^{n-1} + \dots + a_n.$$

We call $p(D)$ a **polynomial differential operator with constant coefficients**. We think of the formal polynomial $p(D)$ as operating on a function $y(x)$, converting it into another function; it is like a black box, in which the function $y(x)$ goes in, and $p(D)y$ (i.e., the left side of (5)) comes out.



Our main goal in this section of the Notes is to develop methods for finding particular solutions to the ODE (5) when $q(x)$ has a special form: an exponential, sine or cosine, x^k , or a product of these. (The function $q(x)$ can also be a sum of such special functions.) These are the most important functions for the standard applications.

The reason for introducing the polynomial operator $p(D)$ is that this allows us to use polynomial algebra to help find the particular solutions. The rest of this chapter of the Notes will illustrate this. Throughout, we let

$$(7) \quad p(D) = D^n + a_1 D^{n-1} + \dots + a_n, \quad a_i \text{ constants.}$$

3. Operator rules.

Our work with these differential operators will be based on several rules they satisfy. In stating these rules, we will always assume that the functions involved are sufficiently differentiable, so that the operators can be applied to them.

Sum rule. If $p(D)$ and $q(D)$ are polynomial operators, then for any (sufficiently differentiable) function u ,

$$(8) \quad [p(D) + q(D)]u = p(D)u + q(D)u .$$

Linearity rule. If u_1 and u_2 are functions, and c_i constants,

$$(9) \quad p(D)(c_1 u_1 + c_2 u_2) = c_1 p(D)u_1 + c_2 p(D)u_2 .$$

The linearity rule is a familiar property of the operator $a D^k$; it extends to sums of these operators, using the sum rule above, thus it is true for operators which are polynomials in D . (It is still true if the coefficients a_i in (7) are not constant, but functions of x .)

Multiplication rule. If $p(D) = g(D)h(D)$, as polynomials in D , then

$$(10) \quad p(D)u = g(D)(h(D)u) .$$

The picture illustrates the meaning of the right side of (10). The property is true when $h(D)$ is the simple operator $a D^k$, essentially because

$$D^m(a D^k u) = a D^{m+k} u;$$

it extends to general polynomial operators $h(D)$ by linearity. Note that a must be a constant; it's false otherwise.

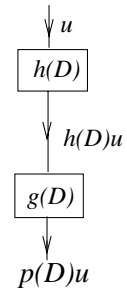
An important corollary of the multiplication property is that *polynomial operators with constant coefficients commute*; i.e., for every function $u(x)$,

$$(11) \quad g(D)(h(D)u) = h(D)(g(D)u) .$$

For as polynomials, $g(D)h(D) = h(D)g(D) = p(D)$, say; therefore by the multiplication rule, both sides of (11) are equal to $p(D)u$, and therefore equal to each other.

The remaining two rules are of a different type, and more concrete: they tell us how polynomial operators behave when applied to exponential functions and products involving exponential functions.

Substitution rule.



$$(12) \quad p(D)e^{ax} = p(a)e^{ax}$$

Proof. We have, by repeated differentiation,

$$De^{ax} = ae^{ax}, \quad D^2e^{ax} = a^2e^{ax}, \quad \dots, \quad D^ke^{ax} = a^ke^{ax};$$

therefore,

$$(D^n + c_1D^{n-1} + \dots + c_n)e^{ax} = (a^n + c_1a^{n-1} + \dots + c_n)e^{ax},$$

which is the substitution rule (12). \square

The exponential-shift rule This handles expressions such as x^ke^{ax} and $x^k \sin ax$.

$$(13) \quad p(D)e^{ax}u = e^{ax}p(D+a)u.$$

Proof. We prove it in successive stages. First, it is true when $p(D) = D$, since by the product rule for differentiation,

$$(14) \quad De^{ax}u(x) = e^{ax}Du(x) + ae^{ax}u(x) = e^{ax}(D+a)u(x).$$

To show the rule is true for D^k , we apply (14) to D repeatedly:

$$\begin{aligned} D^2e^{ax}u &= D(De^{ax}u) = D(e^{ax}(D+a)u) && \text{by (14);} \\ &= e^{ax}(D+a)((D+a)u), && \text{by (14);} \\ &= e^{ax}(D+a)^2u, && \text{by (10).} \end{aligned}$$

In the same way,

$$\begin{aligned} D^3e^{ax}u &= D(D^2e^{ax}u) = D(e^{ax}(D+a)^2u) && \text{by the above;} \\ &= e^{ax}(D+a)((D+a)^2u), && \text{by (14);} \\ &= e^{ax}(D+a)^3u, && \text{by (10),} \end{aligned}$$

and so on. This shows that (13) is true for an operator of the form D^k . To show it is true for a general operator

$$p(D) = D^n + a_1D^{n-1} + \dots + a_n,$$

we write (13) for each $D^k(e^{ax}u)$, multiply both sides by the coefficient a_k , and add up the resulting equations for the different values of k . \square

Remark on complex numbers. By Notes C. (20), the formula

$$(*) \quad D(c e^{ax}) = ca e^{ax}$$

remains true even when c and a are complex numbers; therefore the rules and arguments above remain valid even when the exponents and coefficients are complex. We illustrate.

Example 1. Find $D^3e^{-x} \sin x$.

Solution using the exponential-shift rule. Using (13) and the binomial theorem,

$$\begin{aligned} D^3e^{-x} \sin x &= e^{-x}(D-1)^3 \sin x = e^{-x}(D^3 - 3D^2 + 3D - 1) \sin x \\ &= e^{-x}(2 \cos x + 2 \sin x), \end{aligned}$$

since $D^2 \sin x = -\sin x$, and $D^3 \sin x = -\cos x$.

Solution using the substitution rule. Write $e^{-x} \sin x = \operatorname{Im} e^{(-1+i)x}$. We have

$$\begin{aligned} D^3e^{(-1+i)x} &= (-1+i)^3e^{(-1+i)x}, && \text{by (12) and (*);} \\ &= (2+2i)e^{-x}(\cos x + i \sin x), \end{aligned}$$

by the binomial theorem and Euler's formula. To get the answer we take the imaginary part: $e^{-x}(2 \cos x + 2 \sin x)$.

4. Finding particular solutions to inhomogeneous equations.

We begin by using the previous operator rules to find particular solutions to inhomogeneous polynomial ODE's with constant coefficients, where the right hand side is a real or complex exponential; this includes also the case where it is a sine or cosine function.

Exponential-input Theorem. Let $p(D)$ be a polynomial operator with constant coefficients, and $p^{(s)}$ its s -th derivative. Then

$$(15) \quad p(D)y = e^{ax}, \quad \text{where } a \text{ is real or complex}$$

has the particular solution

$$(16) \quad y_p = \frac{e^{ax}}{p(a)}, \quad \text{if } p(a) \neq 0;$$

$$(17) \quad y_p = \frac{x^s e^{ax}}{p^{(s)}(a)}, \quad \text{if } a \text{ is an } s\text{-fold zero}^1 \text{ of } p.$$

Note that (16) is just the special case of (17) when $s = 0$. Before proving the theorem, we give two examples; the first illustrates again the usefulness of complex exponentials.

Example 2. Find a particular solution to $(D^2 - D + 1)y = e^{2x} \cos x$.

Solution. We write $e^{2x} \cos x = \operatorname{Re}(e^{(2+i)x})$, so the corresponding complex equation is

$$(D^2 - D + 1)\tilde{y} = e^{(2+i)x},$$

and our desired y_p will then be $\operatorname{Re}(\tilde{y}_p)$. Using (16), we calculate

$$p(2+i) = (2+i)^2 - (2+i) + 1 = 2+3i, \quad \text{from which}$$

$$\tilde{y}_p = \frac{1}{2+3i} e^{(2+i)x}, \quad \text{by (16);}$$

$$= \frac{2-3i}{13} e^{2x}(\cos x + i \sin x); \quad \text{thus}$$

$$\operatorname{Re}(\tilde{y}_p) = \frac{2}{13} e^{2x} \cos x + \frac{3}{13} e^{2x} \sin x, \quad \text{our desired particular solution.}$$

Example 3. Find a particular solution to $y'' + 4y' + 4y = e^{-2t}$.

Solution. Here $p(D) = D^2 + 4D + 4 = (D+2)^2$, which has -2 as a double root; using (17), we have $p''(-2) = 2$, so that

$$y_p = \frac{t^2 e^{-2t}}{2}.$$

Proof of the Exponential-input Theorem.

That (16) is a particular solution to (15) follows immediately by using the linearity rule (9) and the substitution rule (12):

$$p(D)y_p = p(D)\frac{e^{ax}}{p(a)} = \frac{1}{p(a)}p(D)e^{ax} = \frac{p(a)e^{ax}}{p(a)} = e^{ax}.$$

¹John Lewis communicated this useful formula.

For the more general case (17), we begin by noting that to say the polynomial $p(D)$ has the number a as an s -fold zero is the same as saying $p(D)$ has a factorization

$$(18) \quad p(D) = q(D)(D - a)^s, \quad q(a) \neq 0.$$

We will first prove that (18) implies

$$(19) \quad p^{(s)}(a) = q(a) s! .$$

To prove this, let k be the degree of $q(D)$, and write it in powers of $(D - a)$:

$$(20) \quad \begin{aligned} q(D) &= q(a) + c_1(D - a) + \dots + c_k(D - a)^k; \quad \text{then} \\ p(D) &= q(a)(D - a)^s + c_1(D - a)^{s+1} + \dots + c_k(D - a)^{s+k}; \\ p^{(s)}(D) &= q(a) s! + \text{positive powers of } D - a; \end{aligned}$$

substituting a for D on both sides proves (19). \square

Using (19), we can now prove (17) easily using the exponential-shift rule (13). We have

$$\begin{aligned} p(D) \frac{e^{ax} x^s}{p^{(s)}(a)} &= \frac{e^{ax}}{p^{(s)}(a)} p(D + a)x^s, && \text{by linearity and (13);} \\ &= \frac{e^{ax}}{p^{(s)}(a)} q(D + a) D^s x^s, && \text{by (18);} \\ &= \frac{e^{ax}}{q(a) s!} q(D + a) s!, && \text{by (19);} \\ &= \frac{e^{ax}}{q(a) s!} q(a) s! = e^{ax}, \end{aligned}$$

where the last line follows from (20), since $s!$ is a constant:

$$q(D + a)s! = (q(a) + c_1D + \dots + c_kD^k) s! = q(a)s! .$$

Polynomial Input: The Method of Undetermined Coefficients.

Let $r(x)$ be a polynomial of degree k ; we assume the ODE $p(D)y = q(x)$ has as input

$$(21) \quad q(x) = r(x), \quad p(0) \neq 0; \quad \text{or more generally, } q(x) = e^{ax} r(x), \quad p(a) \neq 0.$$

(Here a can be complex; when $a = 0$ in (21), we get the pure polynomial case on the left.)

The method is to assume a particular solution of the form $y_p = e^{ax}h(x)$, where $h(x)$ is a polynomial of degree k with unknown (“undetermined”) coefficients, and then to find the coefficients by substituting y_p into the ODE. It’s important to do the work systematically; follow the format given in the following example, and in the solutions to the exercises.

Example 5. Find a particular solution y_p to $y'' + 3y' + 4y = 4x^2 - 2x$.

Solution. Our trial solution is $y_p = Ax^2 + Bx + C$; we format the work as follows. The lines show the successive derivatives; multiply each line by the factor given in the ODE, and add the equations, collecting like powers of x as you go. The fourth line shows the result; the sum on the left takes into account that y_p is supposed to be a particular solution to the given ODE.

$$\begin{array}{r} \times 4 \quad y_p = Ax^2 + Bx + C \\ \times 3 \quad y'_p = \quad 2Ax + B \\ \quad \quad y''_p = \quad \quad 2A \\ 4x^2 - 2x = (4A)x^2 + (4B + 6A)x + (4C + 3B + 2A). \end{array}$$

Equating like powers of x in the last line gives the three equations

$$4A = 4, \quad 4B + 6A = -2, \quad 4C + 3B + 2A = 0;$$

solving them in order gives $A = 1$, $B = -2$, $C = 1$, so that $y_p = x^2 - 2x + 1$.

Example 6. Find a particular solution y_p to $y'' + y' - 4y = e^{-x}(1 - 8x^2)$.

Solution. Here the trial solution is $y_p = e^{-x}u_p$, where $u_p = Ax^2 + Bx + C$.

The polynomial operator in the ODE is $p(D) = D^2 + D - 4$; note that $p(-1) \neq 0$, so our choice of trial solution is justified. Substituting y_p into the ODE and using the exponential-shift rule enables us to get rid of the e^{-x} factor:

$$p(D)y_p = p(D)e^{-x}u_p = e^{-x}p(D-1)u_p = e^{-x}(1 - 8x^2),$$

so that after canceling the e^{-x} on both sides, we get the ODE satisfied by u_p :

$$(22) \quad p(D-1)u_p = 1 - 8x^2; \quad \text{or} \quad (D^2 - D - 4)u_p = 1 - 8x^2,$$

since $p(D-1) = (D-1)^2 + (D-1) - 4 = D^2 - D - 4$.

From this point on, finding u_p as a polynomial solution to the ODE on the right of (22) is done just as in Example 5 using the method of undetermined coefficients; the answer is

$$u_p = 2x^2 - x + 1, \quad \text{so that} \quad y_p = e^{-x}(2x^2 - x + 1).$$

In the previous examples, $p(a) \neq 0$; if $p(a) = 0$, then the trial solution must be altered by multiplying each term in it by a suitable power of x . The book gives the details; briefly, the terms in the trial solution should all be multiplied by the smallest power x^r for which none of the resulting products occur in the complementary solution y_c , i.e., are solutions of the associated homogeneous ODE. Your book gives examples; we won't take this up here.

5. Higher order homogeneous linear ODE's with constant coefficients.

As before, we write the equation in operator form:

$$(23) \quad (D^n + a_1D^{n-1} + \dots + a_n)y = 0,$$

and define its **characteristic equation** or *auxiliary equation* to be

$$(24) \quad p(r) = r^n + a_1r^{n-1} + \dots + a_n = 0.$$

We investigate to see if e^{rx} is a solution to (23), for some real or complex r . According to the substitution rule (12),

$$p(D)e^{rx} = 0 \quad \Leftrightarrow \quad p(r)e^{rx} = 0 \quad \Leftrightarrow \quad p(r) = 0.$$

Therefore

$$(25) \quad e^{rx} \text{ is a solution to (7)} \quad \Leftrightarrow \quad r \text{ is a root of its characteristic equation (16).}$$

Thus, to the real root r_i of (16) corresponds the solution $e^{r_i x}$.

Since the coefficients of $p(r) = 0$ are real, its complex roots occur in pairs which are conjugate complex numbers. Just as for the second-order equation, to the pair of complex conjugate roots $a \pm ib$ correspond the complex solution (we use the root $a + ib$)

$$e^{(a+ib)x} = e^{ax}(\cos bx + i \sin bx),$$

whose real and imaginary parts

$$(26) \quad e^{ax} \cos bx \quad \text{and} \quad e^{ax} \sin bx$$

are solutions to the ODE (23).

If there are n distinct roots to the characteristic equation $p(r) = 0$, (there cannot be more since it is an equation of degree n), we get according to the above analysis n real solutions y_1, y_2, \dots, y_n to the ODE (23), and they can be shown to be linearly independent. Thus the the complete solution y_h to the ODE can be written down immediately, in the form:

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n .$$

Suppose now a real root r_1 of the characteristic equation (24) is a k -fold root, i.e., the characteristic polynomial $p(r)$ can be factored as

$$(27) \quad p(r) = (r - r_1)^k g(r), \quad \text{where } g(r_1) \neq 0 .$$

We shall prove in the theorem below that corresponding to this k -fold root there are k linearly independent solutions to the ODE (23), namely:

$$(28) \quad e^{r_1 x}, \quad x e^{r_1 x}, \quad x^2 e^{r_1 x}, \quad \dots, \quad x^{k-1} e^{r_1 x} .$$

(Note the analogy with the second order case that you have studied already.)

Theorem. *If a is a k -fold root of the characteristic equation $p(r) = 0$, then the k functions in (28) are solutions to the differential equation $p(D)y = 0$.*

Proof. According to our hypothesis about the characteristic equation, $p(r)$ has $(r-a)^k$ as a factor; denoting by $g(r)$ the other factor, we can write

$$p(r) = g(r)(r-a)^k ,$$

which implies that

$$(29) \quad p(D) = g(D)(D-a)^k .$$

Therefore, for $i = 0, 1, \dots, k-1$, we have

$$\begin{aligned} p(D)x^i e^{ax} &= g(D)(D-a)^k x^i e^{ax} \\ &= g(D)((D-a)^k x^i e^{ax}), && \text{by the multiplication rule,} \\ &= g(D)(e^{ax} D^k x^i), && \text{by the exponential-shift rule,} \\ &= g(D)(e^{ax} \cdot 0), && \text{since } D^k x^i = 0 \text{ if } k > i; \\ &= 0, \end{aligned}$$

which shows that all the functions of (20) solve the equation. □

If r_1 is real, the solutions (28) give k linearly independent real solutions to the ODE (23).

In the same way, if $a + ib$ and $a - ib$ are k -fold conjugate complex roots of the characteristic equation, then (28) gives k complex solutions, the real and imaginary parts of which give $2k$ linearly independent solutions to (23):

$$e^{ax} \cos bx, e^{ax} \sin bx, xe^{ax} \cos bx, xe^{ax} \sin bx, \dots, x^{k-1}e^{ax} \cos bx, x^{k-1}e^{ax} \sin bx.$$

Example 6. Write the general solution to $(D + 1)(D - 2)^2(D^2 + 2D + 2)^2y = 0$.

Solution. The characteristic equation is

$$p(r) = (r + 1)(r - 2)^2(r^2 + 2r + 2)^2 = 0.$$

By the quadratic formula, the roots of $r^2 + 2r + 2 = 0$ are $r = -1 \pm i$, so we get

$$y = c_1e^{-x} + c_2e^{2x} + c_3xe^{2x} + e^{-x}(c_4 \cos x + c_5 \sin x + c_6x \cos x + c_7x \sin x)$$

as the general solution to the differential equation. □

As you can see, if the linear homogeneous ODE has constant coefficients, then the work of solving $p(D)y = 0$ is reduced to finding the roots of the characteristic equation. This is “just” a problem in algebra, but a far from trivial one. There are formulas for the roots if the degree $n \leq 4$, but of them only the quadratic formula ($n = 2$) is practical. Beyond that are various methods for special equations and general techniques for approximating the roots. Calculation of roots is mostly done by computer algebra programs nowadays.

This being said, you should still be able to do the sort of root-finding described in Notes C, as illustrated by the next example.

Example 7. Solve: a) $y^{(4)} + 8y'' + 16y = 0$ b) $y^{(4)} - 8y'' + 16y = 0$

Solution. The factorizations of the respective characteristic equations are

$$(r^2 + 4)^2 = 0 \quad \text{and} \quad (r^2 - 4)^2 = (r - 2)^2(r + 2)^2 = 0.$$

Thus the first equation has the double complex root $2i$, whereas the second has the double real roots 2 and -2 . This leads to the respective general solutions

$$y = (c_1 + c_2x) \cos 2x + (c_3 + c_4x) \sin 2x \quad \text{and} \quad y = (c_1 + c_2x)e^{2x} + (c_3 + c_4x)e^{-2x}.$$

6. Justification of the method of undetermined coefficients.

As a last example of the use of these operator methods, we use operators to show where the method of undetermined coefficients comes from. This is the method which assumes the trial particular solution will be a linear combination of certain functions, and finds what the correct coefficients are. It only works when the inhomogeneous term in the ODE (23) (i.e., the term on the right-hand side) is a sum of terms having a special form: each must be the product of an exponential, sin or cos, and a power of x (some of these factors can be missing).

Question: What’s so special about these functions?

Answer: They are the sort of functions which appear as solutions to some linear homogeneous ODE with constant coefficients.

With this general principle in mind, it will be easiest to understand why the method of undetermined coefficients works by looking at a typical example.

Example 8. Show that

$$(30) \quad (D - 1)(D - 2)y = \sin 2x.$$

has a particular solution of the form

$$y_p = c_1 \cos 2x + c_2 \sin 2x .$$

Solution. Since $\sin 2x$, the right-hand side of (30), is a solution to $(D^2 + 4)y = 0$, i.e.,

$$(D^2 + 4) \sin 2x = 0.$$

we operate on both sides of (30) by the operator $D^2 + 4$; using the multiplication rule for operators with constant coefficients, we get (using y_p instead of y)

$$(31) \quad (D^2 + 4)(D - 1)(D - 2)y_p = 0.$$

This means that y_p is one of the solutions to the *homogeneous* equation (31). But we know its general solution: y_p must be a function of the form

$$(32) \quad y_p = c_1 \cos 2x + c_2 \sin 2x + c_3 e^x + c_4 e^{2x} .$$

Now, in (32) we can drop the last two terms, since they contribute nothing: they are part of the complementary solution to (30), i.e., the solution to the associated homogeneous equation. Therefore they need not be included in the particular solution. Put another way, when the operator $(D - 1)(D - 2)$ is applied to (32), the last two terms give zero, and therefore don't help in finding a particular solution to (30).

Our conclusion therefore is that there is a particular solution to (30) of the form

$$y_p = c_1 \cos 2x + c_2 \sin 2x .$$

Here is another example, where one of the inhomogeneous terms is a solution to the associated homogeneous equation, i.e., is part of the complementary function.

Example 9. Find the form of a particular solution to

$$(33) \quad (D - 1)^2 y_p = e^x.$$

Solution. Since the right-hand side is a solution to $(D - 1)y = 0$, we just apply the operator $D - 1$ to both sides of (33), getting

$$(D - 1)^3 y_p = 0.$$

Thus y_p must be of the form

$$y_p = e^x(c_1 + c_2 x + c_3 x^2).$$

But the first two terms can be dropped, since they are already part of the complementary solution to (33); we conclude there must be a particular solution of the form

$$y_p = c_3 x^2 e^x .$$

Exercises: Section 2F